Central Weighted Essential NonOscillatory schemes

Rainer Grauer
Institute for Theoretical Physics I
Ruhr-Universität-Bochum

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1. Semi-discrete central schemes, CWENO

Nessyahu and Tadmor (1990)
Kurganov and Levy (2000) SIAM J. Sci. Comput. 22, 1461

surface plot (2048^2)

with grids (4096^2)

2. Why central schemes?

- no (aproximate) Riemann solver necessary
- dimension by dimension approach makes sence
- high order
- monotone, WENO, TVD depends on the reconstruction
- **easy** for complex problems

Lax-Friedrich

$$u_i^{n+1} = \frac{1}{2}(u_{i+1}^n + u_{i-1}^n) - \frac{dt}{2dx}(f(u_{i+1}^n) - f(u_{i-1}^n)$$

$\Longrightarrow$ dissipation $= \dfrac{(\Delta x)^2}{2\Delta t}$

useless, since

i) high dissipation — need high order

ii) dissipation depends on timestep — need semi-discrete scheme

3. Details

First, consider a 1D conservation law:

$$\partial_t u(x,t) + \partial_x f(u(x,t)) = 0$$

3.1. Fully discrete third order scheme

cell averages

$$\bar{u}_j^n \equiv \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t^n) dx,$$

$\Longrightarrow$

$$\bar{u}_j^{n+1} = \bar{u}_j^n - \frac{1}{\Delta x} \int_{t^n}^{t^{n+1}} \left[f(u(x_{j+1/2}, \tau)) - f(u(x_{j-1/2}, \tau)) \right] d\tau$$

piecewise polynomial reconstruction from the cell averages

$$u(x, t^n) \approx \tilde{u}(x, t^n) = \sum_j P_j(x) \chi_{[x_{j-1/2}, x_{j+1/2}]}.$$

third order scheme: non-oscillatory parabolic reconstruction approximated function $\tilde{u}(x,t^n)$ discontinous at the cell boundaries $x_{j+1/2}$.

different limits $u^{n,+}_{j+1/2}$, $u^{n,-}_{j+1/2}$

$$u^{n,+}_{j+1/2} = P_{j+1}(x_{j+1/2},t^n), \;\; u^{n,-}_{j+1/2} = P_j(x_{j+1/2},t^n),$$

upper bound for the propagation speed of the discontinuities

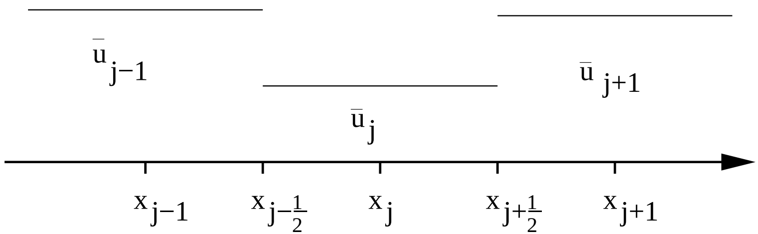
$$a^n_{j+1/2} = \max_{u\in(u^{n,-}_{j+1/2},u^{n,+}_{j+1/2})} \mathsf{abs}\left(\frac{\partial f}{\partial u}(u)\right),$$

$\Longrightarrow$ non-smooth region limited to

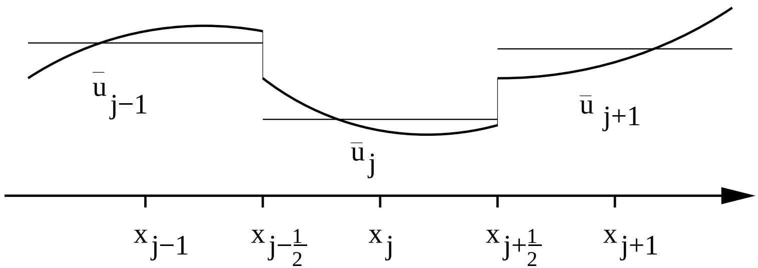
$$x^n_{j+1/2,l} \equiv x_{j+1/2} - a^n_{j+1/2}\Delta t, \;\; x^n_{j+1/2,r} \equiv x_{j+1/2} + a^n_{j+1/2}\Delta t,$$

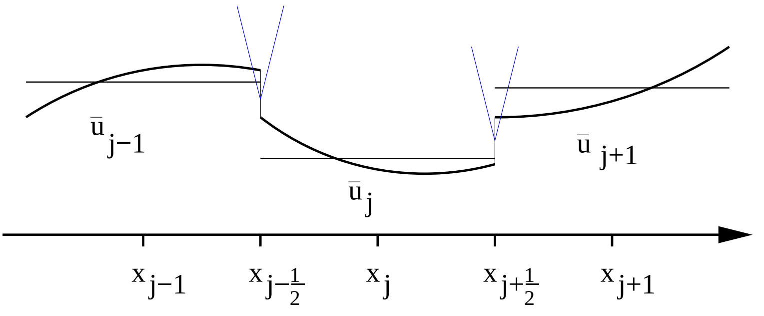
integrate smooth and non-smooth regions independently in time:
new cell averages $\bar{w}^{n+1}_j$ and $\bar{w}^{n+1}_{j+1/2}$ at time t^{n+1} on a non-uniformly spaced, twofold oversampled grid.

$\bar{u}^{n+1}_j$ follows from the $\bar{w}^{n+1}_j$ by polynomial reconstruction

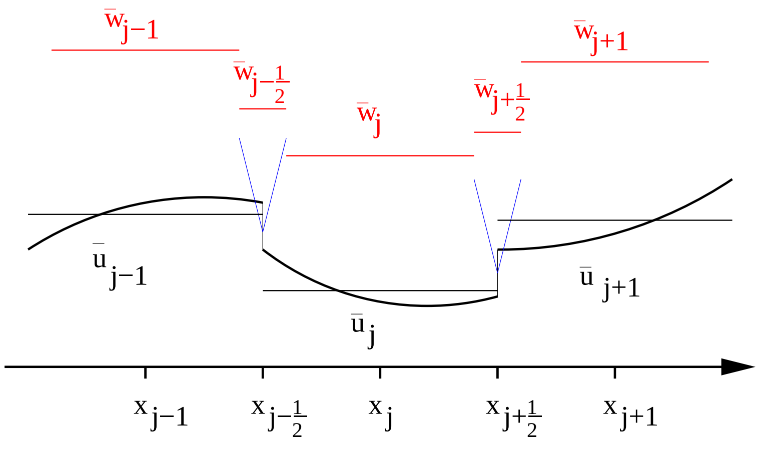


ū j−1
ū j
ū j+1
x j−1
x j−1/2
x j
x j+1/2
x j+1

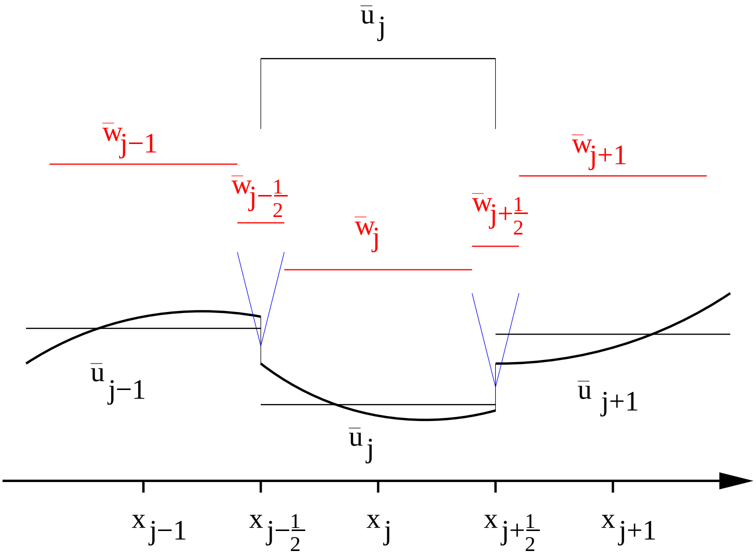

$\bar{u}_{j-1}$
$\bar{u}_j$
$\bar{u}_{j+1}$
x_{j-1}
$x_{j-\frac{1}{2}}$
x_j
$x_{j+\frac{1}{2}}$
x_{j+1}



$\bar{u}_{j-1}$
$\bar{u}_{j+1}$
$\bar{u}_j$
x_{j-1}
$x_{j-\frac{1}{2}}$
x_j
$x_{j+\frac{1}{2}}$
x_{j+1}



$\bar{w}_{j-1}$
$\bar{w}_{j-\frac{1}{2}}$
$\bar{w}_j$
$\bar{w}_{j+\frac{1}{2}}$
$\bar{w}_{j+1}$
$\bar{u}_{j-1}$
$\bar{u}_j$
$\bar{u}_{j+1}$
x_{j-1}
$x_{j-\frac{1}{2}}$
x_j
$x_{j+\frac{1}{2}}$
x_{j+1}



$\bar{u}_j$
$\bar{w}_{j-1}$
$\bar{w}_{j+1}$
$\bar{w}_{j-\frac{1}{2}}$
$\bar{w}_j$
$\bar{w}_{j+\frac{1}{2}}$
$\bar{u}_{j-1}$
$\bar{u}_{j+1}$
$\bar{u}_j$
x_{j-1}
$x_{j-\frac{1}{2}}$
x_j
$x_{j+\frac{1}{2}}$
x_{j+1}

Which steps are actually performed?
Assume, we have the second order polynomial reconstruction:

$$P_j(x,t^n) = A_j + B_j(x-x_j) + \frac{1}{2}C_j(x-x_j)^2$$

A_j, B_j and C_j are determined using the given cell averages $\{u_j^n\}$. Details later.

Integrating over the non-smooth and smooth regions provides us with the non-uniform cell averages $\bar{w}_{j+1/2}^{n+1}$ and $\bar{w}_j^{n+1}$ at time t^{n+1}, respectively:

$$\begin{aligned}
\bar{w}_{j+1/2}^{n+1} &= \frac{A_j + A_{j+1}}{2} + \frac{\Delta x - a_{j+1/2}^n \Delta t}{4}(B_j - B_{j+1}) \\
&\quad + \left(\frac{\Delta x^2}{16} - \frac{a_{j+1/2}^n \Delta t \Delta x}{8} + \frac{(a_{j+1/2}^n \Delta t)^2}{12}\right)(C_j + C_{j+1}) \\
&\quad - \frac{1}{2a_{j+1/2}^n \Delta t}\left\{\int_{t^n}^{t^{n+1}} \left[f(\tilde{u}(x_{j+1/2,r}^n,\tau)) - f(\tilde{u}(x_{j+1/2,l}^n,\tau))\right] d\tau\right\} \\
\bar{w}_j^{n+1} &= A_j + \frac{\Delta t}{2}(a_{j-1/2}^n - a_{j+1/2}^n)B_j \\
&\quad + \left[\frac{\Delta x^2}{24} - \frac{\Delta t \Delta x}{12}(a_{j-1/2}^n + a_{j+1/2}^n) + \frac{\Delta t^2}{6}\left((a_{j-1/2}^n)^2 - a_{j-1/2}^n a_{j+1/2}^n + (a_{j+1/2}^n)^2\right)\right] C_j \\
&\quad - \frac{1}{\Delta x - \Delta t(a_{j-1/2}^n + a_{j+1/2}^n)}\left\{\int_{t^n}^{t^{n+1}} \left[f(\tilde{u}(x_{j+1/2,l}^n,\tau)) - f(\tilde{u}(x_{j-1/2,r}^n,\tau))\right] d\tau\right\}
\end{aligned}$$

Project the non-uniform, twofold oversampled cell averages $\{\bar{w}_j^{n+1}, \bar{w}_{j+1/2}^{n+1}\}$ back onto the original uniform grid $\{\bar{u}_j^{n+1}\}$.

Constant reconstruction in smooth region is sufficient

$$\begin{aligned}
\tilde{w}^{n+1}(x) &= \sum_j \tilde{w}_{j+1/2}^{n+1}(x)\chi_{[x_{j+1/2,l}^n, x_{j+1/2,r}^n]}(x) + \\
&\quad \sum_j \tilde{w}_j^{n+1}(x)\chi_{[x_{j-1/2,r}^n x_{j+1/2,l}^n]} \\
\tilde{w}_{j+1/2}^{n+1}(x) &= \tilde{A}_{j+1/2} + \tilde{B}_{j+1/2}(x - x_{j+1/2}) + \frac{1}{2}\tilde{C}_{j+1/2}(x - x_{j+1/2})^2, \\
\tilde{w}_j^{n+1}(x) &= \bar{w}_j^{n+1},
\end{aligned}$$

The new cell averages $\bar{u}_j^{n+1}$ can then be expressed as follows:

$$\begin{aligned}
\bar{u}_j^{n+1} &= \frac{1}{\Delta x}\left[\int_{x_{j-1/2}}^{x_{j-1/2,r}^n} \tilde{w}_{j-1/2}^{n+1}(x)dx + \int_{x_{j-1/2,r}^n}^{x_{j+1/2,l}^n} \tilde{w}_j^{n+1}(x)dx + \int_{x_{j+1/2,l}^n}^{x_{j+1/2}} \tilde{w}_{j+1/2}^{n+1}(x)dx\right] \\
&= \lambda a_{j-1/2}^n \tilde{A}_{j-1/2} + \left[1 - \lambda(a_{j-1/2}^n + a_{j+1/2}^n)\right] \bar{w}_j^{n+1} + \lambda a_{j+1/2}^n \tilde{A}_{j+1/2} \\
&\quad + \frac{\lambda \Delta t}{2}\left((a_{j-1/2}^n)^2 \tilde{B}_{j-1/2} - (a_{j+1/2}^n)^2 \tilde{B}_{j+1/2}\right) \\
&\quad + \frac{\lambda (\Delta t)^2}{6}\left((a_{j-1/2}^n)^3 \tilde{C}_{j-1/2} + (a_{j+1/2}^n)^3 \tilde{C}_{j+1/2}\right),
\end{aligned}$$

where $\lambda = \Delta t / \Delta x$.

3.2. Transition to the third order semi-discrete scheme

Now consider the limit of $\Delta t \longrightarrow 0$ to derive the semi-discrete scheme:

$$\frac{d}{dt}\bar{u}_j(t) = \lim_{\Delta t \longrightarrow 0} \frac{\bar{u}_j^{n+1} - \bar{u}_j^n}{\Delta t}.$$

Result:

$$\begin{aligned} \frac{d\bar{u}_j}{dt} &= -\frac{1}{2\Delta x}\left[f(u^+_{j+1/2}(t)) + f(u^-_{j+1/2}(t)) - f(u^+_{j-1/2}(t)) + f(u^-_{j-1/2}(t))\right] \\ &\quad + \frac{a_{j+1/2}(t)}{2\Delta x}\left[u^+_{j+1/2}(t) - u^-_{j+1/2}(t)\right] + \frac{a_{j-1/2}(t)}{2\Delta x}\left[u^+_{j-1/2}(t) - u^-_{j-1/2}(t)\right] \end{aligned}$$

3.3. Weighted ENO reconstruction

In each cell we need to reconstruct a polynomial approximation P_{EXACT} to the real solution from the known cell averages.
We use a second order ansatz for the polynomial

$$\begin{aligned} P_{\text{EXACT}}(x,y) &= u^n_{ij} + u^n_{ij,x}(x-x_j) + \frac{1}{2}u^n_{ij,xx}(x-x_j)^2 + \\ &\quad u^n_{ij,y}(y-y_j) + \frac{1}{2}u^n_{ij,yy}(y-y_j)^2 \end{aligned}$$

The five coefficients

$$u_{ij}^n \ , \ \ u_{ij,x}^n \ , \ \ u_{ij,xx}^n \ , \ \ u_{ij,y}^n \ , \ \ u_{ij,yy}^n$$

are determined by requiring the polynomial to conserve the cell averages

$$\bar{u}_{mn}^n \ \text{ for } \ (m,n) \in \{(i,j),(i+1,j),(i-1,j),(i,j+1),(i,j-1)\}.$$

The coefficients are given by

$$\begin{aligned}
u_{ij}^n &= \bar{u}_{ij}^n - \frac{1}{24}(\bar{u}_{i+1,j}^n - 2\bar{u}_{ij}^n + \bar{u}_{i-1,j}^n) - \frac{1}{24}(\bar{u}_{i,j+1}^n - 2\bar{u}_{ij}^n + \bar{u}_{i,j-1}^n), \\
u_{ij,x}^n &= \frac{\bar{u}_{i+1,j}^n - \bar{u}_{i-1,j}^n}{2\Delta x}, \quad u_{ij,y}^n = \frac{\bar{u}_{i,j+1}^n - \bar{u}_{i,j-1}^n}{2\Delta x} \\
u_{ij,xx}^n &= \frac{\bar{u}_{i+1,j}^n - 2\bar{u}_{i,j}^n + \bar{u}_{i-1,j}^n}{\Delta x^2}, \quad u_{ij,yy}^n = \frac{\bar{u}_{i,j+1}^n - 2\bar{u}_{i,j}^n + \bar{u}_{i,j-1}^n}{\Delta y^2}.
\end{aligned}$$

P_{EXACT} is a a good approximation to the real function $u(x,y;t^n)$

BUT it does not provide non-oscillatory behavior.

Solution: Weighted ENO

Discuss now construction of the interpolating polynomial for the x-direction.
Dimension-by-dimension approach

In each cell reconstruct quadratic polynomial as a convex combination of three polynomials

$$P_j(x) = w_L P_L(x) + w_R P_R(x) + w_C P_C(x),$$

with positive weights $w_i > 0$ and $\sum_i w_i = 1$, where $i \in \{L, R, C\}$.

The polynomials $P_L(x), P_R(x)$ correspond to left and right one-sided linear reconstructions, uniquely determined by requiring them to conserve the one-sided cell averages:

$$\begin{aligned} \bar{u}_{ij} &= \int_{(i-1/2)\Delta x}^{(i+1/2)\Delta x} P_R(x)\,dx \text{ and } \bar{u}_{i,j+1} = \int_{(i+1/2)\Delta x}^{(i+3/2)\Delta x} P_R(x)\,dx \\ \bar{u}_{ij} &= \int_{(i-1/2)\Delta x}^{(i+1/2)\Delta x} P_L(x)\,dx \text{ and } \bar{u}_{i,j-1} = \int_{(i-3/2)\Delta x}^{(i-1/2)\Delta x} P_L(x)\,dx \end{aligned}$$

The polynomial $P_C(x)$ is determined by

$$P_{\text{EXACT}}(x, y = y_j) = c_L P_L(x) + c_R P_R(x) + (1 - c_L - c_R) P_C(x)$$

Every symmetric selection of the coefficients $c_L = c_R$ will provide third-order accuracy.

Choosing $c_L = c_r = 1/4$ we obtain the polynomial $P_C(x)$ as

$$\begin{aligned} P_C(x) &= \bar{u}^n_{ij} - \frac{1}{12}(\bar{u}^n_{i+1,j} - 2\bar{u}^n_{ij} + \bar{u}^n_{i-1,j}) - \frac{1}{12}(\bar{u}^n_{i,j+1} - 2\bar{u}^n_{ij} + \bar{u}^n_{i,j-1}) \\ &+ \frac{\bar{u}^n_{i+1,j} - \bar{u}^n_{i-1,j}}{2\Delta x}(x - x_j) + \frac{1}{2}\frac{\bar{u}^n_{i+1,j} - 2\bar{u}^n_{i,j} + \bar{u}^n_{i-1,j}}{\Delta x^2}(x - x_j)^2 \end{aligned}$$

The weights w_i are used to automatically adapt the reconstruction to the smoothness of the solution. In smooth regions, they select the third-order reconstruction to provide maximum precision, whereas in the presence of discontinuities they switch to a one-sided reconstruction to guarantee the essentially non-oscillatory behavior.
The weights are taken as

$$w_i = \frac{\alpha_i}{\sum_m \alpha_m}\ , \ \text{where}\ \alpha_i = \frac{c_i}{(\epsilon + IS_i)^p}\ , \quad i, m \in \{c, R.L\}$$

$$C_L = C_R = 1/4\ , \ \ C_C = 1/2$$

Smoothness indicator

$$IS_l = (\bar{u}^n_j - \bar{u}^n_{j-1})\ , \quad IS_l = (\bar{u}^n_{j+1} - \bar{u}^n_j)$$

$$IS_C = \frac{13}{3}(\bar{u}^n_{j+1} - 2\bar{u}^n_j + \bar{u}^n_{j-1})^2 + \frac{1}{4}(\bar{u}^n_{j+1} - \bar{u}^n_{j-1})^2$$

4. Results from the papers

5. Example code

6. Compact differencing for the Helmholtz equation

6.1. 2D Euler

6.2. 2D electron MHD

6.3. timestepping algorithm

7. The adaptive mesh treatment

7.1. Interpolation to finer grids

7.2. Updating the coarser grids

7.3. Scale dependent CFL condition

8. Numerical Results