Resistive MHD simulations of the interaction of the solar wind with magnetized planets

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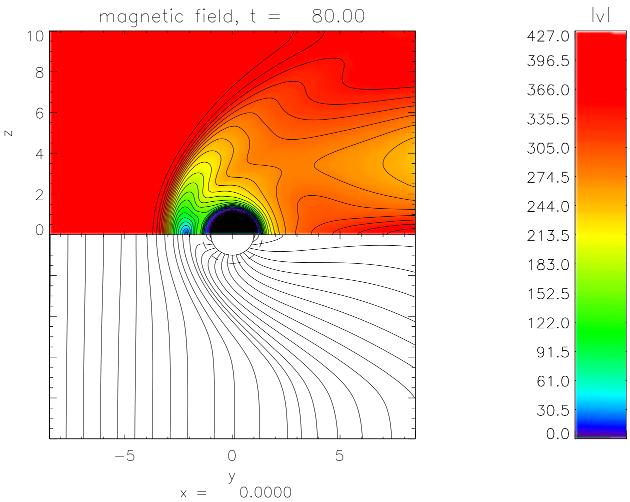
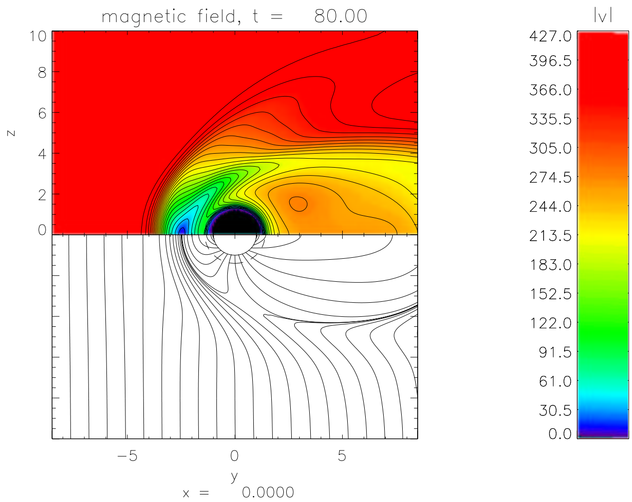
- Example: Interaction of the solar wind with Mercury:
  - colour: flow velocity of the solar wind in km/s
  - field lines: magnetic field (a superposition of the interplanetary magnetic field (IMF) and Mercury's dipole field)



(southward IMF)



(northward IMF)

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  - *Guido Birk (LMU München)*
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Basic equations of resistive MHD with mass source:

$$\boxed{\begin{aligned} \frac{\partial \rho}{\partial t} &= -\vec{\nabla}\cdot(\rho\vec{v}) + \dot{\rho} \\ \frac{\partial \rho\vec{v}}{\partial t} &= -\vec{\nabla}\cdot(\rho\vec{v}\,\vec{v}) - \vec{\nabla}P + \vec{j}\times\vec{B} + \dot{\rho}\vec{v}_q \\ \frac{\partial \vec{B}}{\partial t} &= -\vec{\nabla}\times\vec{E} = \vec{\nabla}\times(\vec{v}\times\vec{B} - \eta\vec{j}) \\ \frac{\partial \rho\varepsilon}{\partial t} &= -\vec{\nabla}\cdot(\rho\varepsilon\vec{v}) - P\vec{\nabla}\cdot\vec{v} + \eta\vec{j}^2 + \frac{\dot{\rho}}{\rho}P \end{aligned}}$$

- Maxwell equations:

$$\rightarrow \vec{j}\times\vec{B} = \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})\times\vec{B} = \frac{1}{\mu_0}\Big(\vec{\nabla}\cdot(\vec{B}\,\vec{B}) + (\vec{\nabla}\cdot\vec{B})\vec{B} - \frac{1}{2}\vec{\nabla}(\vec{B}\cdot\vec{B})\Big)$$

$$\rightarrow \frac{\partial \rho\vec{v}}{\partial t} = -\vec{\nabla}\cdot\Big(\big(P + \vec{B}^{\,2}/2\mu_0\big)\mathbf{1} + \big(\rho\vec{v}\,\vec{v} - (\vec{B}\,\vec{B})/\mu_0\big)\Big)$$

- Ideal gas: $\rho\varepsilon = 1/(\gamma-1)P$ $\rightarrow$ equation for P alone:

$$\frac{1}{\gamma-1}\frac{\partial P}{\partial t} = -\frac{1}{\gamma-1}\vec{\nabla}\cdot(P\vec{v}) - P\vec{\nabla}\cdot\vec{v} + \eta\vec{j}^2 + \frac{\dot{\rho}}{\rho}P$$

- The resistivity can depend on the current density (anomaleous resistivity)
- Normalization leads to $P \rightarrow P/2$ and $\eta \rightarrow \eta/S$ (Lundquist number)
- Further simplification by introducing the function $u := (P/2)^{\frac{1}{\gamma}}$

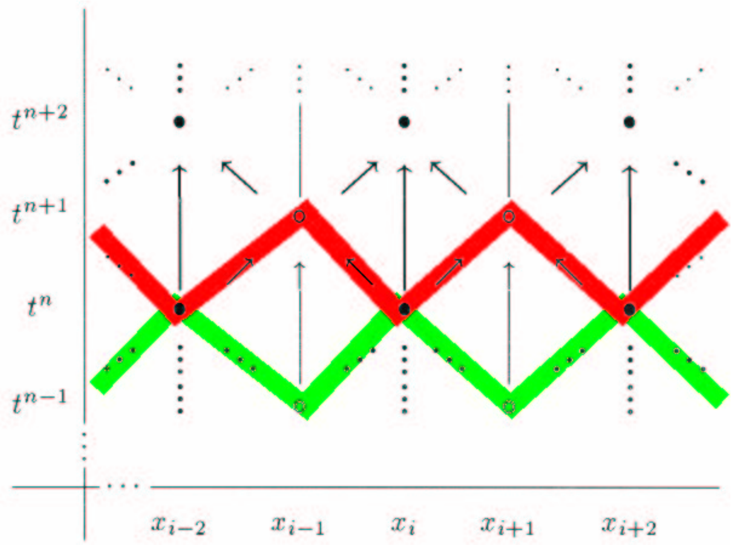
$\rightarrow$ Normalized equations (with $\vec{s} := \rho\vec{v}$):

$$\boxed{\begin{aligned} \frac{\partial \rho}{\partial t} &= -\vec{\nabla}\cdot\vec{s} + \dot{\rho} \\ \frac{\partial \vec{s}}{\partial t} &= -\vec{\nabla}\cdot\Big(\vec{s}\,\vec{v} + u^\gamma\mathbf{1}\Big) + \vec{j}\times\vec{B} + \dot{\rho}\vec{v}_q \\ &= -\vec{\nabla}\cdot\Big(\vec{s}\,\vec{v} - \vec{B}\,\vec{B} + (u^\gamma + \tfrac{1}{2}\vec{B}^{\,2})\mathbf{1}\Big) + \dot{\rho}\vec{v}_q \\ \frac{\partial \vec{B}}{\partial t} &= \quad \vec{\nabla}\times\Big(\vec{v}\times\vec{B}\Big) + \frac{1}{S}\Big(\eta\Delta\vec{B} - \vec{\nabla}\eta\times\vec{j}\Big) \\ \frac{\partial u}{\partial t} &= -\vec{\nabla}\cdot\Big(u\vec{v}\Big) + \frac{\gamma-1}{\gamma}u^{1-\gamma}\Big(\frac{\dot{\rho}}{\rho}u^\gamma + \eta\vec{j}^2\Big) \end{aligned}}$$

The numerical method:

- The balance equations are hyperbolic → ***Leapfrog*** scheme
- The induction equation is parabolic → ***Dufort-Frankel*** scheme
- One-dimensional example:
  - time steps: $t_n := t_0 + n\Delta t$
  - grid: $x_i := x_{min} + i\Delta x$, with: $x_1 = x_{min}$, $x_N = x_{max}$
- Both schemes are of second order in space and time

→ The numerical grid is divided into two staggered subgrids:



- There are only grid points with n+i = even*):
  - grid "∘": n = odd, i = odd
  - grid "•": n = even, i = even

  → ***no grid points with n+i = odd : n = odd/even, i = even/odd!***
  - in the code: the time is not labeled, but included implicitly, e.g.
    ① index = i → t=n (•), index=i±1 → t=n−1 (∘) (=grid before step 1†))
    ② index = i → t=n (•), index=i±1 → t=n+1 (∘) (=grid after step 1)
- All grid points are known at $t = 0$ (labeled then as grid "•")

  → a half integration step initializes grid "∘"

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*) or odd, depending on the definition of the grid †) cf. next page

The integration schemes:

- Abbrevations: $u_i^n := u(x = x_i, t = t_n)$, $F_i^n := F(u_i^n)$

- ***The leapfrog scheme:***
  - simplest example:

$$\frac{\partial u}{\partial t} = -\frac{\partial F(u)}{\partial x}$$

  - step 1 (computes grid ∘):

$$u_{i+1}^{n+1} = u_{i+1}^{n-1} - \frac{\Delta t}{\Delta x}\left(F_{i+2}^n - F_i^n\right) \quad \text{and} \quad u_{i-1}^{n+1} = u_{i-1}^{n-1} - \frac{\Delta t}{\Delta x}\left(F_i^n - F_{i-2}^n\right)$$

  - step 2 (computes grid •):

$$u_i^{n+2} = u_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1}^{n+1} - F_{i-1}^{n+1}\right)$$

- ***The Dufort-Frankel scheme:***
  - simplest example:

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}$$

  - the leapfrog scheme is unstable in this case

    (and the point u_{i+1}^n does not exist; in the code: u_{i+1} stands for u_{i+1}^{n-1})

  → time average of the Laplace term:
  - scheme demonstrated here only for the first step:

$$u_{i+1}^{n+1} = u_{i+1}^{n-1} + \kappa \frac{2\Delta t}{\Delta x^2}\left(u_{i+2}^n - \underbrace{\left(u_{i+1}^{n+1} + u_{i+1}^{n-1}\right)}_{\text{average}} + u_i^n\right)$$

  → semi-implicit scheme, equation can be solved for u_{i+1}^{n+1}:

$$\rightarrow \qquad u_{i+1}^{n+1} = \left(u_{i+1}^{n-1} + \kappa \frac{2\Delta t}{\Delta x^2}\left(u_{i+2}^n - u_{i+1}^{n-1} + u_i^n\right)\right) \Big/ \left(1 + \kappa \frac{2\Delta t}{\Delta x^2}\right)$$

Numerical realization of the solar wind interaction with a planet:

- Coordinates:
  - $\hat{y}$ flow direction, $\hat{z}$ main direction of the IMF (S→N), $\hat{x} = \hat{y} \times \hat{z}$
  - the planet (with radius R) is located at the origin (only for $t > 0$, see below)
- Initial state:
  - all quantities are homogeneous, $\vec{v} = v_{SW}\hat{y}$
  - the planet does not occur in the initial state, but ...
- ... is introduced by a time-dependent function f which slows down the plasma:

$$f(x,y,z,t) = \kappa\left(\left(\frac{t}{\tau}\right)^2\right)\cos^2\left(\frac{\pi}{2}\kappa(\tilde{r})\right), \text{ with: } \kappa(x) := \max(\min(x,1),0),$$

  - $\tilde{r} := (r - R)/\Delta r$ and $r := \sqrt{x^2 + y^2 + z^2}$
  - Δr is a "smoothness parameter" (typical value: $\Delta r = 0.4$)

- The source terms:
  - mass loading takes place within a sphere of radius $R_A > R$ (the "atmosphere")
  - expression: $\dot{\rho} = Q_0 \exp\Big(-(r' - R)/((R_A - R)\ln(\varepsilon))\Big)$, with $r' = \max(r, R)$

  → $\dot{\rho}$ has the value $\dot{\rho} = Q_0\varepsilon$ at $r = R_A$
  - velocity: $\vec{v}_q$ is the velocity of the newly generated particles:

    e.g. $\vec{v}_q = \vec{v}$ for pickup ions or $\vec{v}_q = v_0\hat{r}$ for a plasma source ($\hat{r}$: radius vector)

- Numerical realization of an intrinsic dipole field:
  - e.g. for magnetic moment in z-direction: $\vec{B} = B_0(3xz, 3yz, 2z^2 - x^2 - y^2)/r^5$
  - problems: field diverges for $r \to 0$, steep gradient

  → define a radius $r_0 < R$ (typical value: $r_0 = 0.6R$)

  → replace $1/r^5$ by $g(r) = (1/{r_0}^5)((n+5)(r/r_0)^{n-1} - (n+4)(r/r_0)^n)$ (e.g. $n = 8$)

  → keep the magnetic field fixed for $r \leq r_0$ (do not integrate)

  → subtract quantities vanishing analytically in the initial state: $\vec{\jmath} \to \vec{\jmath} - \vec{\jmath}_{initial}$, same for $\Delta\vec{B}$ (if necessary also for the rhs. of the momentum balance eqn.)

Boundary conditions in upstream direction

- all quanties are kept fixed (with a smooth transition to the interior)

Method for the other boundaries: Consider lower (left) boundary in 1D:

- x_1: numerical boundary (for derivatives), x_2: physical boundary

→ x_1 and x_2 belong to different grids → compute u_1 with u_3 and u_5:

$$u_1 = ku_3 - au_3'(x_3 - x_1)$$

→ special cases: $a = 0$ and $k = (-)1$ (anti)symmetry, $a > 0$: extrapolation

- extrapolation: need "wrong" values u_2 and u_4 for the computation of u_3'

  → use average of u_2' and u_4' instead, where u_2' contains the (unknown) value u_1:

$$\rightarrow u_1 = ku_3 - a\Big(\frac{x_4 - x_3}{x_4 - x_2}u_2' + \frac{x_3 - x_2}{x_4 - x_2}u_4'\Big)(x_3 - x_1)$$
$$= ku_3 - a\Big(\frac{x_4 - x_3}{x_4 - x_2}\frac{u_3 - u_1}{x_3 - x_1} + \frac{x_3 - x_2}{x_4 - x_2}\frac{u_5 - u_3}{x_5 - x_3}\Big)(x_3 - x_1)$$

  → semi-implicit, for "exact" extrapolation ($a = 1$, $k = 1$) this leads to:

$$u_1 = \frac{x_5 - x_3 + x_3 - x_1}{x_5 - x_3}u_3 - \frac{x_3 - x_1}{x_5 - x_3}u_5 \;\rightarrow\; \text{this means: resulting } u_2' = u_4'!$$

→ problems when perturbations move towards the boundary (→ oszillations)

New method: extrapolate u_2' and make use of the fact that the derivatives (but not necessarily the values) for both grids are continuous:

$$u_2' = u_3' - (x_3 - x_2)\Big(\frac{u_4' - u_3'}{x_4 - x_3}\Big)$$

- introduce a "damping factor" $0 \leq \alpha \leq 1$ (to avoid overshooting) and write:

$$u_1 = u_3 - \alpha u_2'(x_3 - x_1) \ \ (u_2' \text{ is now known and doesn't contain } u_1)$$

→ results in an expression $u_1 = C_1 u_2 + C_2 u_2 + C_3 u_4 + C_4 u_5$ with:

$$C_2 = \alpha\frac{x_3 - x_1}{x_4 - x_3}, \quad C_3 = 1 - C_2\,\alpha\frac{x_3 - x_2}{x_5 - x_3}, \quad C_4 = -C_2, \quad C_5 = 1 - C_3$$

→ couples the grids (helps against the odd-even instability)

- if perturbations propagating to the boundary still cause problems, u_2 can be computed with the same method (with $u_3 - u_6$) before u_1 is computed

DENISIS (Dust Electron Neutral Ion Self-consistent Integration Scheme):

- 4-fluid model, where the dust can be replaced by a 2^{nd} ion species
- Basic equations for each species α (α = e, i, n, or d):

$$\begin{aligned}
\frac{\partial \rho_\alpha}{\partial t} &= -\vec{\nabla} \cdot (\rho_\alpha \vec{v}_\alpha) + Q_\alpha^C \\
\frac{\partial \rho_\alpha \vec{v}_\alpha}{\partial t} &= -\vec{\nabla} \cdot (\rho_\alpha \vec{v}_\alpha \, \vec{v}_\alpha) - \vec{\nabla} P_\alpha + q_\alpha (\vec{E} + \vec{v}_\alpha \times \vec{B}) + \vec{Q}_\alpha^S + \vec{F}_\alpha^S \\
\frac{\partial \vec{B}}{\partial t} &= -\vec{\nabla} \times \vec{E} \\
\frac{1}{\gamma_\alpha - 1} \frac{\partial P \alpha}{\partial t} &= -\frac{1}{\gamma_\alpha - 1} \vec{\nabla} \cdot (P_\alpha \vec{v}_\alpha) - P_\alpha \vec{\nabla} \cdot \vec{v_\alpha} + Q_\alpha^E + F_\alpha^E
\end{aligned}$$

- Source terms: Q: interaction between the species, F: external sources
- Charge $q_\alpha = \sigma_\alpha e z_\alpha$ where $\sigma_{e/d} = -1$, $\sigma_i = +1$ (z_α: charge number)
- Basic asumptions (only necessary for the electrons):
  - quasineutrality $\rightarrow \rho_e = m_e (z_i \rho_i / m_i - z_d \rho_d / m_d)$
  - electrons inertialess: $d(\rho_e \vec{v}_e)/dt = 0 \rightarrow$ equation for the electric field

  $\rightarrow$ the electron velocity is computed from: $\vec{\nabla} \times \vec{B} = \mu_0 \vec{\jmath} = \mu_0 \sum_\alpha q_\alpha n_\alpha \vec{v}_\alpha$

- Numerical realization of the source terms (contain "wrong" grid points u_{i+1}^n):

$$\frac{\partial u}{\partial t} = -\frac{\partial F(u)}{\partial x} + \alpha u$$

  $\rightarrow$ new method with 3 steps:

  ① replace u_{i+1}^t by its spatial average and perform a half step:

$$u_{i+1}^n = u_{i+1}^{n-1} - \frac{\Delta t}{2\Delta x}\left(u_{i+2}^n - u_i^n\right) + \frac{\alpha \Delta t}{2}\left(u_{i+2}^n + u_i^n\right)$$

  ② substitute the averaged source term by the newly computed one:

$$\tilde{u}_{i+1}^n = u_{i+1}^n - \alpha \Delta t \left(\frac{1}{2}\left(u_{i+2}^n + u_i^n\right) + u_{i+1}^n\right)$$

  ③ perform the second half of the time step with the new source term:

$$u_{i+1}^{n+1} = \tilde{u}_{i+1}^n - \frac{\Delta t}{2\Delta x}\left(u_{i+2}^n - u_i^n\right) + \alpha \Delta t u_{i+1}^n$$

The Dust-MHD equations:

- Continuity equations:

$$\rho_e = m_e(z_i\rho_i/m_i - z_d\rho_d/m_d) \quad \text{(QN)}$$
$$\frac{\partial\rho_i}{\partial t} = \vec{\nabla}\cdot\rho_i\vec{v}_i + \iota\rho_n - \xi\rho_i\rho_e$$
$$\frac{\partial\rho_d}{\partial t} = \vec{\nabla}\cdot\rho_d\vec{v}_d$$
$$\frac{\partial\rho_n}{\partial t} = \vec{\nabla}\cdot\rho_n\vec{v}_n - \iota\rho_n + \xi\rho_i\rho_e$$

(ι: ionisation frequency, ξ: recombination frequency)

- Momentum balance equations:
  - $d(\rho_e\vec{v}_e)/dt = 0$ gives the electric field:

$$\vec{E} = -\frac{m_e}{e\rho_e}\vec{\nabla}P_e - \vec{v}_e\times\vec{B} + \frac{m_e}{e\rho_e}(\vec{F}_e^S + \vec{Q}_e^S)$$

  - $\vec{v}_e$ follows from the current density:

$$\vec{v}_e = \frac{m_e\rho_i}{m_i\rho_e}z_i\vec{v}_i - \frac{m_e\rho_d}{m_d\rho_e}z_d\vec{v}_d - \frac{m_e}{\mu_0 e\rho_e}\vec{\nabla}\times\vec{B}$$

  - equations for the other species:

$$\frac{\partial\rho_i\vec{v}_i}{\partial t} = -\vec{\nabla}\cdot(\rho_i\vec{v}_i\,\vec{v}_i) - \vec{\nabla}P_i + q_i(\vec{E}+\vec{v}_i\times\vec{B}) + \rho_i\vec{g} + \vec{\Gamma}_i + \iota\rho_i\vec{v}_n - \xi\rho_i\rho_e\vec{v}_i$$
$$- \nu_{ei}\rho_e(\vec{v}_e - \vec{v}_i) - \nu_{ni}\rho_n(\vec{v}_n - \vec{v}_i) - \nu_{di}\rho_d(\vec{v}_d - \vec{v}_i)$$
$$\frac{\partial\rho_n\vec{v}_n}{\partial t} = -\vec{\nabla}\cdot(\rho_n\vec{v}_n\,\vec{v}_n) - \vec{\nabla}P_n + \rho_n\vec{g} + \vec{\Gamma}_n - \iota\rho_i\vec{v}_n + \xi\rho_i\rho_e\vec{v}_i$$
$$- \nu_{en}\rho_e(\vec{v}_e - \vec{v}_n) - \nu_{in}\rho_i(\vec{v}_i - \vec{v}_n) - \nu_{dn}\rho_d(\vec{v}_d - \vec{v}_n)$$
$$\frac{\partial\rho_d\vec{v}_d}{\partial t} = -\vec{\nabla}\cdot(\rho_d\vec{v}_d\,\vec{v}_d) - \vec{\nabla}P_d + q_d(\vec{E}+\vec{v}_d\times\vec{B}) + \rho_d\vec{g} + \vec{\Gamma}_d$$
$$- \nu_{ed}\rho_e(\vec{v}_e - \vec{v}_d) - \nu_{id}\rho_i(\vec{v}_i - \vec{v}_d) - \nu_{nd}\rho_n(\vec{v}_n - \vec{v}_d)$$

($\vec{g}$: external gravitation, $\vec{\Gamma}_\alpha$: radiation pressure, $\nu_{\alpha\beta}$: collision frequency, with symmetry condition: $\rho_\alpha\nu_{\alpha\beta} = \rho_\beta\nu_{\beta\alpha}$)

- Induction equation:

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{v}_e \times \vec{B}) + \frac{m_e}{e}\frac{1}{\rho_e^2}\vec{\nabla} P_e \times \vec{\nabla}\rho_e - \frac{m_e}{e}\vec{\nabla} \times \left(\frac{1}{\rho_e}\vec{\Gamma}_e\right)$$
$$+ \frac{m_e}{e}\vec{\nabla} \times \left(\frac{1}{\rho_e}\big(\nu_{ie}\rho_i(\vec{v}_i - \vec{v}_e) + \nu_{ne}\rho_n(\vec{v}_n - \vec{v}_e) + \nu_{de}\rho_d(\vec{v}_d - \vec{v}_e)\big)\right)$$

$\rightarrow$ with $\vec{v}_e$ inserted:

$$\frac{\partial \vec{B}}{\partial t} = \frac{m_e\rho_i}{m_i\rho_e} z_i \vec{\nabla} \times (\vec{v}_i \times \vec{B}) - \frac{m_e\rho_d}{m_d\rho_e} z_d \vec{\nabla} \times (\vec{v}_d \times \vec{B})$$
$$- \frac{m_e}{\mu_0 e\rho_e}\vec{\nabla} \times \big((\vec{\nabla} \times \vec{B}) \times \vec{B}\big) + \frac{m_e}{e}\frac{1}{\rho_e^2}\vec{\nabla} P_e \times \vec{\nabla}\rho_e - \frac{m_e}{e}\vec{\nabla} \times \left(\frac{1}{\rho_e}\vec{\Gamma}_e\right)$$
$$+ \frac{m_e}{e}\vec{\nabla} \times \Bigg(\frac{1}{\rho_e}\Big(\nu_{ie}\rho_i\big((1 - \frac{m_e\rho_i}{m_i\rho_e} z_i)\vec{v}_i + \frac{m_e\rho_d}{m_d\rho_e} z_d\vec{v}_d + \frac{m_e}{\mu_0 e\rho_e}\vec{\nabla} \times \vec{B}\big)$$
$$+ \nu_{ne}\rho_n\big(\vec{v}_n - \frac{m_e\rho_i}{m_i\rho_e} z_i\vec{v}_i + \frac{m_e\rho_d}{m_d\rho_e} z_d\vec{v}_d + \frac{m_e}{\mu_0 e\rho_e}\vec{\nabla} \times \vec{B}\big)$$
$$+ \nu_{de}\rho_d\big((1 + \frac{m_e\rho_d}{m_d\rho_e} z_d)\vec{v}_d - \frac{m_e\rho_i}{m_i\rho_e} z_i\vec{v}_i + \frac{m_e}{\mu_0 e\rho_e}\vec{\nabla} \times \vec{B}\big)\Big)\Bigg)$$

$\rightarrow$ includes the Hall term

$\rightarrow$ a magnetic field may be generated just by collisions, without any seed field

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- Conservation laws:
  - ***Mass:*** $M = \sum_\alpha \int \rho_\alpha d^3 r = 0 \rightarrow$ $\boxed{\sum_\alpha Q_\alpha^C = 0}$
  - ***Momentum:*** $\vec{p} = \int(\sum_\alpha \rho_\alpha \vec{v}_\alpha + \vec{\nabla} \cdot \mathbf{M} - (\sigma_{el}\vec{E} + \vec{j} \times \vec{B}))d^3 r = 0$

    ($\mathbf{M} = \varepsilon_0 \vec{E}\,\vec{E} + (1/\mu_0)\vec{B}\,\vec{B} - \frac{1}{2}(\varepsilon_0 \vec{E}^{\,2} + ((1/\mu_0)\vec{B}^{\,2})\mathbf{1}$: Maxwellian stress tensor)

    $\rightarrow$ $\boxed{\sum_\alpha \vec{Q}_\alpha^S = 0}$
  - ***Energy:*** $\mathcal{E} = \int(\sum_\alpha(\rho_\alpha\varepsilon_\alpha + \frac{1}{2}\rho_\alpha\vec{v}_\alpha^2) + \varepsilon_0\vec{E}^{\,2} + (1/\mu_0)\vec{B}^{\,2})d^3 r = 0$

    $\rightarrow$ $\boxed{\sum_\alpha(Q_\alpha^E + \vec{v}_\alpha \cdot \vec{Q}_\alpha^S - \frac{1}{2}\vec{v}_\alpha^2 Q_\alpha^C) = 0}$
  - conditions for the mass and the momentum satisfied
  - additional terms in the energy equations

- Energy equations:

$$\frac{1}{\gamma_e-1}\frac{\partial Pe}{\partial t} = -\frac{1}{\gamma_e-1}\vec{\nabla}\cdot(P_e\vec{v}_e) - P_e\vec{\nabla}\cdot\vec{v_e} + \mathcal{L}_e^r$$
$$+ \frac{\gamma_n}{\gamma_n-1}\frac{m_e}{m_e+m_i}\frac{k_BT_n}{m_n}\iota\rho_n - \frac{\gamma_e}{\gamma_e-1}\frac{k_BT_e}{m_e}\xi\rho_i\rho_e$$
$$+ \frac{m_i}{m_i+m_e}\nu_{ie}\rho_i(\vec{v}_i-\vec{v}_e)^2 + \frac{m_n}{m_n+m_e}\nu_{ne}\rho_n(\vec{v}_n-\vec{v}_e)^2$$
$$+ \frac{m_d}{m_d+m_e}\nu_{de}\rho_d(\vec{v}_d-\vec{v}_e)^2 - 2\frac{\rho_e\nu_{ei}}{m_e+m_i}\left(\frac{k_BT_e}{\gamma_e-1}-\frac{k_BT_i}{\gamma_i-1}\right)^2$$
$$- 2\frac{\rho_e\nu_{en}}{m_e+m_n}\left(\frac{k_BT_e}{\gamma_e-1}-\frac{k_BT_n}{\gamma_n-1}\right)^2 - 2\frac{\rho_e\nu_{ed}}{m_e+m_d}\left(\frac{k_BT_e}{\gamma_e-1}-\frac{k_BT_d}{\gamma_d-1}\right)^2$$

$$\frac{1}{\gamma_i-1}\frac{\partial Pi}{\partial t} = -\frac{1}{\gamma_i-1}\vec{\nabla}\cdot(P_i\vec{v}_i) - P_i\vec{\nabla}\cdot\vec{v_i} + \mathcal{L}_i^r + \frac{\gamma_n}{\gamma_n-1}\frac{m_i}{m_e+m_i}\frac{k_BT_n}{m_n}\iota\rho_n$$
$$- \frac{\gamma_i}{\gamma_i-1}\frac{k_BT_i}{m_i}\xi\rho_i\rho_e + \frac{m_i}{2(m_n+m_i)}(\iota\rho_n+\xi\rho_i\rho_e)(\vec{v}_i-\vec{v}_n)^2$$
$$+ \frac{m_e}{m_e+m_i}\nu_{ei}\rho_e(\vec{v}_e-\vec{v}_i)^2 + \frac{m_n}{m_n+m_i}\nu_{ni}\rho_n(\vec{v}_n-\vec{v}_i)^2$$
$$+ \frac{m_d}{m_d+m_i}\nu_{di}\rho_d(\vec{v}_d-\vec{v}_i)^2 - 2\frac{\rho_i\nu_{ie}}{m_i+m_e}\left(\frac{k_BT_i}{\gamma_i-1}-\frac{k_BT_e}{\gamma_e-1}\right)^2$$
$$- 2\frac{\rho_i\nu_{in}}{m_i+m_n}\left(\frac{k_BT_i}{\gamma_i-1}-\frac{k_BT_n}{\gamma_n-1}\right)^2 - 2\frac{\rho_i\nu_{id}}{m_i+m_d}\left(\frac{k_BT_i}{\gamma_i-1}-\frac{k_BT_d}{\gamma_d-1}\right)^2$$

$$\frac{1}{\gamma_n-1}\frac{\partial Pn}{\partial t} = -\frac{1}{\gamma_n-1}\vec{\nabla}\cdot(P_n\vec{v}_n) - P_n\vec{\nabla}\cdot\vec{v_n} + \mathcal{L}_n^r - \frac{\gamma_n}{\gamma_n-1}\frac{k_BT_n}{m_n}\iota\rho_n$$
$$+ \left(\frac{\gamma_e}{\gamma_e-1}\frac{k_BT_e}{m_e} + \frac{\gamma_i}{\gamma_i-1}\frac{k_BT_i}{m_e}\right)\xi\rho_i\rho_e + \frac{m_n}{2(m_i+m_n)}(\iota\rho_n+\xi\rho_i\rho_e)(\vec{v}_i-\vec{v}_n)^2$$
$$+ \frac{m_e}{m_e+m_n}\nu_{en}\rho_e(\vec{v}_e-\vec{v}_n)^2 + \frac{m_i}{m_i+m_n}\nu_{in}\rho_i(\vec{v}_i-\vec{v}_n)^2$$
$$+ \frac{m_d}{m_d+m_n}\nu_{dn}\rho_n(\vec{v}_d-\vec{v}_n)^2 - 2\frac{\rho_n\nu_{ne}}{m_n+m_e}\left(\frac{k_BT_n}{\gamma_n-1}-\frac{k_BT_e}{\gamma_e-1}\right)^2$$
$$- 2\frac{\rho_n\nu_{ni}}{m_n+m_i}\left(\frac{k_BT_n}{\gamma_n-1}-\frac{k_BT_i}{\gamma_i-1}\right)^2 - 2\frac{\rho_n\nu_{nd}}{m_n+m_d}\left(\frac{k_BT_n}{\gamma_n-1}-\frac{k_BT_d}{\gamma_d-1}\right)^2$$

$$\frac{1}{\gamma_d-1}\frac{\partial Pd}{\partial t} = -\frac{1}{\gamma_d-1}\vec{\nabla}\cdot(P_d\vec{v}_d) - P_d\vec{\nabla}\cdot\vec{v_d} + \mathcal{L}_d^r$$
$$+ \frac{m_e}{m_e+m_d}\nu_{ed}\rho_e(\vec{v}_e-\vec{v}_d)^2 + \frac{m_i}{m_i+m_d}\nu_{id}\rho_i(\vec{v}_i-\vec{v}_d)^2$$
$$+ \frac{m_n}{m_n+m_d}\nu_{nd}\rho_n(\vec{v}_n-\vec{v}_d)^2 - 2\frac{\rho_d\nu_{de}}{m_d+m_e}\left(\frac{k_BT_d}{\gamma_d-1}-\frac{k_BT_e}{\gamma_e-1}\right)^2$$
$$- 2\frac{\rho_d\nu_{di}}{m_d+m_i}\left(\frac{k_BT_d}{\gamma_d-1}-\frac{k_BT_i}{\gamma_i-1}\right)^2 - 2\frac{\rho_d\nu_{dn}}{m_d+m_n}\left(\frac{k_BT_d}{\gamma_d-1}-\frac{k_BT_n}{\gamma_n-1}\right)^2$$

  - $\mathcal{L}_\alpha^r$ radiative gains, losses