Computing the gravitational potential of extended objects with axial symmetry

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http://www.tp4.rub.de/~jk/science/gravity/gravpot-disk.pdf

1 Basic formula for ring potential

Given a homogeneous, thin ring of radius r_0 and mass m located at height z_0 (in cylindrical coordinates.) The ring's plane is perpendicular to the z axis. Let

$$\begin{aligned} \mathcal{X} &:= [r, z, r_0, z_0] \\ A(\mathcal{X}) &:= \left[(r - r_0)^2 + (z - z_0)^2\right] / r_0^2 \\ &= (\bar{r} - 1)^2 + (\bar{z} - \bar{z}_0)^2 \end{aligned}$$

with normalisations $\bar{r} := r/r_0$ and $\bar{z} := z/r_0$.
Question: What is the resulting gravitational potential at $\mathbf{r} = (r, 0, z)$?
Distance to ring element of length $r_0\, \mathrm{d}\varphi$ located at $\mathbf{r}'$:

$$|\mathbf{r} - \mathbf{r}'| = \left| \begin{pmatrix} r \\ 0 \\ z \end{pmatrix} - \begin{pmatrix} r_0 \cos\varphi \\ r_0 \sin\varphi \\ z_0 \end{pmatrix} \right| = \sqrt{r^2 - 2rr_0 \cos\varphi + r_0^2 + (z - z_0)^2}$$

Due to homogeneity, the ratio of mass element to ring's total mass is

$$\mathrm{d}m/m = \mathrm{d}\varphi/(2\pi)\ .$$

$$\begin{aligned} \Rightarrow \Phi(r_0, r, z) = -\int_{\text{ring}} \frac{G\, \mathrm{d}m}{|\mathbf{r} - \mathbf{r}'|} &= -\frac{Gm}{2\pi} \int_0^{2\pi} \frac{\mathrm{d}\varphi}{\sqrt{r^2 - 2rr_0 \cos\varphi + r_0^2 + z^2}} \\ &= -\frac{Gm}{2\pi r_0} \int_0^{2\pi} \frac{\mathrm{d}\varphi}{\sqrt{\bar{r}^2 - 2\bar{r} \cos\varphi + 1 + \bar{z}^2}} \end{aligned}$$

where $z_0 = 0$ has been assumed without loss of generality.
If $A(\mathcal{X}) \neq 0$, this can be written as (remember $\cos\varphi = 1 - 2\sin^2(\varphi/2)$):

$$\Phi(r_0, r, z) = -\frac{Gm}{r_0\,\sqrt{A(\mathcal{X})}} \times \begin{cases} 1 & : \; r = 0 \vee r_0 = 0 \\ \frac{2}{\pi}\,K\left(\frac{4\bar{r}}{A(\mathcal{X})}\right) & : \; \text{else} \end{cases}$$

where

$$K(k) := \int_0^{\pi/2} \frac{\mathrm{d}\varphi}{\sqrt{1 + k\sin^2\varphi}} \equiv E\left(\sqrt{-k}\right)$$

and $E(\cdot)$ is the complete elliptic integral of the first kind.
If $A(\mathcal{X}) = 0$, then $(r, z) = (r_0, z_0)$, and the integral becomes

$$\frac{Gm}{2\sqrt{2}\pi r_0} \int_0^{2\pi} \frac{\mathrm{d}\varphi}{\sqrt{1 - \cos\varphi}} = \frac{Gm}{8\pi r_0} \int_0^{\pi/2} \frac{\mathrm{d}\varphi}{\sin\varphi}\ ,$$

which diverges due to a singularity at $\varphi = 0$.

2 Potential of annular disk

To find the potential of a thin, annular disk with inner and outer radius aR and R (where $0 \leq a < 1$), we can just integrate the contributions from rings with $r_0 \in [aR, R]$.
Mass of a single ring (previously denoted by m) is $\mathrm{d}M = 2\pi r_0\ \Delta r_0 \Delta z\ \rho$. Mass ratio of infinitisemal ring vs. complete disk ($a = 0$):

$$\frac{\mathrm{d}M}{M} = \frac{2\pi r_0\ \mathrm{d}r_0\ \Delta z\ \rho}{\pi\ R^2\ \Delta z\ \rho} = \frac{2r_0\ \mathrm{d}r_0}{R^2}$$

Potential of thin disk with radius $[aR, R]$ located at $z_0 = 0$:

$$\begin{aligned} \Phi_{\mathrm{D}}(r, z) = \int_{r_0 = aR}^{R} \mathrm{d}\Phi(r_0, r, z) &= -\frac{GM}{2\pi} \int_{aR}^{R} \frac{2r_0}{R^2} \int_0^{2\pi} \frac{\mathrm{d}\varphi\ \mathrm{d}r_0}{\sqrt{r^2 - 2rr_0\cos\varphi + r_0^2 + z^2}} \\ &= -\frac{GM}{\pi R} \int_a^1 \int_0^{2\pi} \frac{x\ \mathrm{d}\varphi\ \mathrm{d}x}{\sqrt{\bar{r}^2 - 2\bar{r}x\cos\varphi + x^2 + \bar{z}^2}} \end{aligned}$$

$$= -\frac{4\ \Phi_0}{\pi} \int\limits_a^1 \frac{x}{\sqrt{(\bar{r}-x)^2+\bar{z}^2}}\ K\left(\frac{4\ \bar{r}x}{(\bar{r}-x)^2+\bar{z}^2}\right) \mathrm{d}x$$

where $x := r_0/R$, $\Phi_0 := GM/R$, and $\bar{r} \equiv r/R$ and $\bar{z} \equiv z/R$ are now normalised to the outer radius R. For the remainder of this text, the bars will be dropped, implying that all lengths are given in units of R. Likewise, potentials and angular velocities are normalized to Φ_0 and $\Omega_0 := \sqrt{GM/R^3}$.
Along the z axis, we have in particular

$$\frac{\Phi_\mathrm{D}(0,z)}{\Phi_0} = -\frac{1}{\pi} \int\limits_a^1 \int\limits_0^{2\pi} \frac{x\ \mathrm{d}\varphi\ \mathrm{d}x}{\sqrt{x^2+z^2}} = -2\left[\sqrt{1+z^2} - \sqrt{a+z^2}\right] \ .$$

3 Potential balancing by rotation

Potentials of other physical origin can added linearly due to superposition. In particular, if a point mass $M_\mathrm{star} = \mu M$ is present at the origin, and one wishes the inner and outer rim (at $z = 0$ and $r \in \{a, 1\}$) to have the same total potential

$$\begin{aligned} \Phi_\mathrm{tot}(r,z) &:= \Phi_\mathrm{disk}(r,z) + \Phi_\mathrm{star}(r,z) + \Phi_\mathrm{rot}(r,z) \\ &= \Phi_\mathrm{D}(r,z) - \frac{\mu}{\sqrt{r^2+z^2}} - \frac{(\Omega\ r)^2}{2} \ , \end{aligned}$$

one can solve

$$\Phi_\mathrm{tot}(a,0) = \Phi_\mathrm{tot}(1,0)$$

for Ω, yielding

$$\Omega = \sqrt{\frac{2}{1+a}\left(\frac{\mu}{a} + \frac{\Phi_\mathrm{D}(1,0) - \Phi_\mathrm{D}(a,0)}{1-a}\right)} \ .$$

4 Potential balancing by variable area density

As an alternative to rigid rotation, one can introduce a variable mass area density (mass per area) $\sigma(r) \equiv s(r)\ M/(\pi R^2)$, such that $s(r)$ is dimensionless. We consider a disk partitioned into N concentric, plane-circular annuli with radial ranges $[r_i, r_{i+1}]$, where $r_i := [a + (1-a)(i/N)]$. (This is a linear mapping $[0, N] \mapsto [a, 1]$).

4.1 Piecewise constant area density

If $s(r)$ is equal to a constant s_i on each annuli, the gravitational potential at radius r within the disk plane due to annulus i is

$$\Phi_i(r) = -G \int\limits_{\text{ann.}i} \frac{\sigma_i \, \mathrm{d}A}{|\mathbf{r} - \mathbf{r}'|} = -\frac{s_i}{\pi} \int\limits_{r_i}^{r_{i+1}} \int\limits_{0}^{2\pi} \frac{x \, \mathrm{d}\varphi \, \mathrm{d}x}{\sqrt{r^2 - 2rx\cos\varphi + x^2}} =: -s_i \; L_i(r) \; .$$

We require the disk's total potential (i.e. of all annuli combined) to be constant at each $r_{i+1/2}$ (at the middle of each annulus):

$$\Phi_\mathrm{c} \stackrel{!}{=} \Phi_\mathrm{tot}(r_{i+1/2}) = -\sum_{j=0}^{N-1} s_j \; L_j(r_{i+1/2}) \quad \forall i$$

This is equivalent to the matrix equation $\Phi_\mathrm{c}\mathbf{u} = \mathcal{A} \; \mathbf{s}$ with

$$\begin{aligned} \mathbf{u} \; &:= \; (-1, \cdots, -1)^\mathrm{T} \\ (\mathcal{A})_{ij} \; &:= \; L_j(r_{i+1/2}) \end{aligned}$$

such that the components of $\mathbf{s}$ are found by inverting $\mathcal{A}$:

$$\mathbf{s} = \Phi_\mathrm{c} \; \mathcal{A}^{-1} \; \mathbf{u} =: \Phi_\mathrm{c} \; \mathbf{p}$$

The normalisation factor Φ_c can be fixed by requiring

$$M(1-a^2) \stackrel{!}{=} \sum_{i=0}^{N-1} \left(\frac{M}{\pi R^2} \; s_i \right) \pi(r_{i+1}^2 - r_i^2) \quad \Leftrightarrow \quad 1 - a^2 = \sum_{i=0}^{N-1} \Phi_\mathrm{c} \; p_i(r_{i+1}^2 - r_i^2) \; ,$$

finally leading to

$$s_i = p_i \; (1-a^2) \left[\sum_{i=0}^{N-1} p_i \left(r_{i+1}^2 - r_i^2 \right) \right]^{-1} \; .$$

It should be noted that this procedure can be used equally well to prescribe any other values at any other radii, simply by changing vector $\mathbf{u}$ to hold the desired values, and matrix $\mathcal{A}$ to be evaluated at the desired radii.

4.2 Piecewise linear area density

We now wish $s(r)$ to have values s_i at radii r_i $(i = 0, \dots, N)$ and be piecewise linear (rather than constant) in between:

$$s(r) = a_i + r b_i \quad \forall r \in [r_i, r_{i+1}]$$

with

$$a_i = \frac{s_i \; r_{i+1} - s_{i+1} \; r_i}{r_{i+1} - r_i} \quad \text{and} \quad b_i = \frac{s_{i+1} - s_i}{r_{i+1} - r_i} \; .$$

The total potential is then

$$\begin{aligned} \Phi_{\text{tot}}(r, z) &= -\int_a^1 \int_0^{2\pi} \frac{s(x) \; x \; \mathrm{d}\varphi \; \mathrm{d}x}{\sqrt{r^2 - 2rx\cos\varphi + x^2 + z^2}} \\ &= -\sum_{i=0}^{N-1} a_i \; L_i(r, z) + b_i \; Q_i(r, z) \end{aligned}$$

where

$$\begin{aligned} L_i(r, z) &:= \frac{1}{\pi} \int_{r_i}^{r_{i+1}} \int_0^{2\pi} \frac{x \; \mathrm{d}\varphi \; \mathrm{d}x}{\sqrt{r^2 - 2rx\cos\varphi + x^2 + z^2}} = \frac{4}{\pi} \int_{r_i}^{r_{i+1}} K\left(-\frac{2rx}{(r-x)^2 + z^2}\right) \frac{x \; \mathrm{d}x}{\sqrt{(r-x)^2 + z^2}} \\ Q_i(r, z) &:= \frac{1}{\pi} \int_{r_i}^{r_{i+1}} \int_0^{2\pi} \frac{x^2 \; \mathrm{d}\varphi \; \mathrm{d}x}{\sqrt{r^2 - 2rx\cos\varphi + x^2 + z^2}} = \frac{4}{\pi} \int_{r_i}^{r_{i+1}} K\left(-\frac{2rx}{(r-x)^2 + z^2}\right) \frac{x^2 \; \mathrm{d}x}{\sqrt{(r-x)^2 + z^2}} \end{aligned}$$

The requirement of constant potential at the interfaces[1] r_j becomes

$$\begin{aligned} \Phi_{\text{c}} \stackrel{!}{=} \Phi_{\text{tot}}(r_j, 0) &= -\sum_{i=0}^{N-1} \left(\frac{s_i \; r_{i+1} - s_{i+1} \; r_i}{r_{i+1} - r_i}\right) L_i(r_j, 0) + \left(\frac{s_{i+1} - s_i}{r_{i+1} - r_i}\right) Q_i(r_j, 0) \\ &= -\sum_{i=0}^{N-1} \left(\frac{r_{i+1} L_i(r_j, 0) - Q_i(r_j, 0)}{r_{i+1} - r_i}\right) s_i + \left(\frac{-r_i L_i(r_j, 0) + Q_i(r_j, 0)}{r_{i+1} - r_i}\right) s_{i+1} \\ &= -\sum_{i=0}^{N} \left(\frac{r_{i+1} L_i(r_j, 0) - Q_i(r_j, 0)}{r_{i+1} - r_i} + \frac{-r_{i-1} L_{i-1}(r_j, 0) + Q_{i-1}(r_j, 0)}{r_i - r_{i-1}}\right) s_i \end{aligned}$$

[1]Since $s(r)$ is now piecewise linear, $N + 1$ coefficients must be fixed, rather than just N as in the picewise constant case. For this reason, the canonical choice for the radii at which to prescribe the potential are the interfaces, not the annuli's central radii.

$$=: \quad -\sum_{i=0}^{N} \mathcal{B}_{ji} s_i = -(\mathcal{B}\ \mathbf{s})_j$$

or, in matrix notation,

$$\Phi_\mathrm{c}\ \mathbf{u} = \mathcal{B}\ \mathbf{s}$$

where L_i and Q_i are identically zero for $i \in \{-1, N\}$. The remaining procedure is identical to the one used in the piecewise constant case, except that the final normalisation is done via

$$\begin{aligned}
(1-a^2)M \quad &\overset{!}{=} \quad \int_{aR}^{R} \sigma(r) \frac{M}{\pi R^2}\ 2\pi\ r\ \mathrm{d}r \\
\Rightarrow \quad (1-a^2) \quad &\overset{!}{=} \quad 2\int_{a}^{1} s(x)\ x\ \mathrm{d}x \quad = \quad 2\sum_{i=0}^{N-1} \int_{r_i}^{r_{i+1}} (a_i + x b_i)\ x\ \mathrm{d}x \\
&= \quad 2\sum_{i=0}^{N-1} \left[\left(\frac{s_i\ r_{i+1} - s_{i+1}\ r_i}{r_{i+1} - r_i} \right) \frac{r_{i+1}^2 - r_i^2}{2} + \left(\frac{s_{i+1} - s_i}{r_{i+1} - r_i} \right) \frac{r_{i+1}^3 - r_i^3}{3} \right] \\
&= \quad \frac{\Phi_\mathrm{c}}{3} \sum_{i=0}^{N-1} (r_{i+1} - r_i) \left[p_i\ (2r_i + r_{i+1}) + p_{i+1}\ (r_i + 2r_{i+1}) \right]
\end{aligned}$$