Large acoustic solitons and double layers in plasmas with two positive ion species

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Large nonlinear acoustic waves are discussed in a plasma made up of cold supersonic and adiabatic subsonic positive ions, in the presence of hot isothermal electrons, with the help of Sagdeev pseudopotential theory. In this model, no solitons are found at the acoustic speed, and no compositional parameter ranges exist where solutions of opposite polarities can coexist. All nonlinear modes are thus super-acoustic, but polarity changes are possible. The upper limits on admissible structure velocities come from different physical arguments, in a strict order when the fractional cool ion density is increased: infinite cold ion compression, warm ion sonic point, positive double layers, negative double layers, and finally, positive double layers again. However, not all ranges exist for all mass and temperature ratios. Whereas the cold and warm ion sonic point limitations are always present over a wide range of mass and temperature ratios, and thus positive polarity solutions can easily be obtained, double layers have a more restricted existence range, specially if polarity changes are sought. © *2011 American Institute of Physics*. [doi:10.1063/1.3579397]

I. INTRODUCTION

Using a Sagdeev pseudopotential formalism[1] where nonlinear structures are stationary in a comoving frame, large acoustic solitary waves and double layers have been studied in quite a number of different plasma models, far too many to cite even a representative selection thereof. For weakly nonlinear solitary waves, Korteweg–de Vries (KdV) theory, usually found through the reductive perturbation technique, leads to structures whose amplitude tends to zero as the wave speed approaches the speed of the sound. The traditional Sagdeev theory requires that the pseudopotential have a maximum at the origin, resulting in super-acoustic solitary waves and these are KdV-like (their amplitudes vanish at the acoustic speed).

However, several recent examples of pseudopotentials for three-component plasma models have been found that yield finite amplitude solitons at the acoustic speed itself.[2–5] Even a simple proton–electron plasma can show such behavior,[6] when the hot electrons have a Cairns nonthermal distribution[7] at a sufficient degree of nonthermality. These results are in contradistinction to conventional wisdom, where the existence of solitary solutions requires that these be super-acoustic and none exist at the acoustic speed itself. If there are solitary structures at the acoustic speed, neighboring Sagdeev potentials then have solitons of both polarities, one KdV-like (its amplitude vanishes at the acoustic speed), the other necessarily being finite.[2–6] The recently found solitary waves and double layers, that have finite amplitude at the acoustic speed are termed non-KdV-like, and for those the Sagdeev pseudopotential has a local maximum on one side only, due to the very topology of the function at the origin, in this case.[3,4]

In a recent paper,[8] we investigated large nonlinear acoustic waves in a four-component plasma, made up of superhot isothermal electrons and protons, and two inertial species with lower thermal velocities, being, respectively, adiabatic and cold. An example of the models covered is one in which the isothermal species are electrons and ions, while the cooler species consist of positive and/or negative dust. Solitary waves can only occur for nonlinear structure velocities smaller than the adiabatic dust thermal velocity, leading to a novel dust-acoustic-like mode based on the interplay between the two dust species.

If the cold and adiabatic dusts are oppositely charged, only solitary waves, having the polarity of the cold dust, exist, their parameter range being limited by infinite compression of the cold dust. However, when the charges of the cold and adiabatic species have the same sign, solitary structures are limited in velocity, and for increasing values of f, the ratio of cold to adiabatic species charge densities, successively by infinite cold dust compression, by encountering the adiabatic dust sonic point and by the occurrence of double layers. The latter have, for lower f values, the same polarity as the cold dust but switch at the high f end to the opposite polarity.

In the light of our recent findings about the role played by the Sagdeev pseudopotential at the acoustic speed,[2–6] we now study a slightly simpler plasma model, comprised of positive cold supersonic and adiabatic subsonic ions, in the presence of hot isothermal electrons. The terminology subsonic (hot) and supersonic (cool) comes from comparing the nonlinear structure velocity V to each species' thermal velocity, and for solitary structures at least one subsonic and one supersonic species is needed.[9,10] Most of our conclusions tally qualitatively with those of the four-component model,[8] but the existence domains in parameter space and the corresponding amplitudes are now delineated in a systematic way, rather than relying solely on numerical experimentation. In particular, there are new polarity switches at the very high f end, i.e., for a dominant cool ion population, with only a small admixture of adiabatic ions.

Before going on, we recall that in a three-fluid plasma one has, in principle, two global acoustic speeds,[11] one in each successive interval between the species' thermal velocities.[12] These acoustic speeds are the minima for the respective soliton velocities. As in our model one of the thermal velocities is zero (cold), and another infinite (inertialess, isothermal electrons), one of the global acoustic speeds is smaller and the other one larger than the thermal velocity of the adiabatic ions. For reasons explained below, we restrict the discussion in the present paper to the case where the global acoustic speed is smaller than the adiabatic ion thermal velocity, so that the adiabatic ions are subsonic, in the fluid dynamical terminology.[9,10] This has a bearing on the way in which the adiabatic ion density is expressed.

When the global acoustic speed is larger than the adiabatic ion thermal velocity, the adiabatic ions are supersonic. For the record, we have also applied the new insight[2–6] to this case, where both positive ion species are supersonic, leaving the hot electrons as the only subsonic species, but found only positive polarity solutions. This agrees qualitatively with analogous dust ion-acoustic soliton studies in plasmas composed of positive dust together with either polytropic electrons and cold ions[13] or κ-distributed electrons and adiabatic ions.[3] In the absence of polarity changes, such plasma regimes merely yield extensions of the standard ion-acoustic solitary waves[1,9,10,14,15] and have therefore not been included in this paper. We also recall that in simple plasmas having two polytropic species, only one soliton polarity is supported, corresponding to the sign of the supersonic species.[10]

Therefore, the possibility of having polarity switches or indeed "coexistence" (where both positive and negative solitons are supported for a given set of plasma parameters) requires either a nonpolytropic subsonic species in a two-component plasma, e.g., by having a Cairns distribution,[6,7] or the addition of at least a third species. Of the latter, there are many examples to be found in the literature, and we only mention in the reference list some immediately relevant papers, namely those homogeneous models that admit coexistence ranges in compositional parameter space for electrostatic solitary modes described by Sagdeev pseudopotential theory.[16–35] Extraneous effects such as beams, inhomogeneities, and variable dust charges have been excluded, as these tend to blur the basic understanding. The quest for plasma models supporting solitons having a polarity opposite to the traditionally expected has been motivated by several space observations.[36–42]

The paper is structured as follows: after the Introduction, we recall in Sec. II some elements of the basic Sagdeev formalism and determine the pseudopotential for the problem at hand. Section III is devoted to a systematic discussion of the existence domains in parameter space, together with the corresponding amplitudes that can be so obtained and illustrated by well chosen Sagdeev pseudopotential curves. Finally, our conclusions are briefly summarized in Sec. IV.

II. FORMALISM AND SAGDEEV PSEUDOPOTENTIALS

We start from a plasma model containing three species that are stationary in an inertial frame: positive cold supersonic (with label c) and adiabatic subsonic ions (labeled a), and hot isothermal electrons (with label e).

Inertia is retained for the two ion species, which introduces two masses in the description, the adiabatic ion mass being used for the normalization of the physical quantities and the cold ion mass expressed through a mass ratio $\mu = m_a/m_c$. On the other hand, there are two species with pressure effects, the subsonic adiabatic ions and the electrons. Again, the thermal effect of the adiabatic ions disappears in the normalization, the electron temperature being expressed through the ratio $\tau = T_a/T_e$.

For physical reasons, we will, in the applications, mostly use $\mu \leqslant 1$, because it seems logical that the warmer ions are lighter than the colder ions, for which temperature and pressure effects could justifiably be omitted. This, however, has no bearing upon the mathematics of our derivation, and we will include illustrations for a case in which $\mu = 10$ is considered. Similarly, the electrons, being described as Boltzmann, are assumed to have the highest temperature, which implies $\tau \leqslant 1$.

In the absence of inertial effects, the dimensionless electron density n_e is given by a standard Boltzmann distribution,

$$n_e = \exp[\tau\varphi], \tag{1}$$

where the number densities of the different species j are normalized by their undisturbed values, denoted where needed by $\tilde{n}_{j0}$, and the electrostatic potential φ has been normalized with respect to T_a/e. Speeds are normalized with respect to the adiabatic thermal velocity c_{ta}, which is formally defined through $c_{ta}^2 = T_a/m_a = 3\tilde{p}_{a0}/\tilde{n}_{a0}m_a$, where $\tilde{p}_{a0}$ is the equilibrium value of the adiabatic species' pressure.

Combining the continuity and momentum equations for the cold ion species, we obtain in the moving frame for the cold ion density that[8]

$$n_c = \left(1 - \frac{2\mu\varphi}{M^2}\right)^{-1/2}. \tag{2}$$

The adiabatic ion "Mach number" (Ref. 9) $M = V/c_{ta}$ involves both V and c_{ta}. We note that M is not the true Mach number, but it nevertheless serves as a measure for the structure speed, and, for the purposes of subsequent parameter discussions, this measure does not vary with f, μ, and τ, as normalizing with respect to the global acoustic speed would. Although V is the (absolute value of the) velocity of the soliton as seen in an inertial frame, we will work in the wave frame and then V corresponds to the undisturbed plasma velocities in this frame, faraway from the soliton, and M is a positive real.

For the adiabatic species we follow our previous discussion,[8] and start from the stationary form of the fluid equations of continuity and momentum, written as

$$\frac{d}{dx}(n_a u_a) = 0, \tag{3}$$

$$u_a \frac{du_a}{dx} + n_a \frac{dn_a}{dx} = -\frac{d\varphi}{dx}, \tag{4}$$

where the fluid velocities u_a have been scaled by c_{ta} and space x by $\sqrt{\epsilon_0 T_a/\tilde{n}_{e0}e^2}$. Time has dropped out of the description when addressing stationary structures in a comoving frame. Finally, we have taken singly charged ions for ease of exposition, but higher charge numbers Z_a and Z_c can easily be incorporated by appropriate changes in the normalization.

With the help of mass (flux) conservation $n_a u_a = M$, obtained from the integration of (3), we integrate (4) to obtain a biquadratic equation for n_a, the correct solution of which yields

$$n_a^2 = \frac{1}{2}\left[M^2 + 1 - 2\varphi + \sqrt{(M^2 + 1 - 2\varphi)^2 - 4M^2}\right]. \tag{5}$$

Here, the plus sign has been chosen in front of the square root, so that for a subsonic species ($M < 1$) the correct limit n_{a0} is obtained at $\varphi = 0$ (Ref. 8). Following the ideas of Ghosh *et al.*,[43] the adiabatic ion density is given by

$$n_a = \frac{1}{2}\left[\sqrt{(1+M)^2 - 2\varphi} + \sqrt{(1-M)^2 - 2\varphi}\right]. \tag{6}$$

Whereas the electron density in (1) is well behaved for all φ, the ion densities given in (2) and (6) encounter limits for $\varphi > 0$, beyond which they are no longer defined and real. These limits are, respectively,

$$\varphi_{\ell c} = \frac{M^2}{2\mu},$$
$$\varphi_{\ell a} = \frac{1}{2}(1-M)^2. \tag{7}$$

We see that $\varphi_{\ell c}$ vanishes at $M = 0$ and increases monotonically with M to $1/2\mu$ at $M = 1$, whereas $\varphi_{\ell a}$ decreases monotonically from $1/2$ at $M = 0$ to zero at $M = 1$. Hence, there is one and only one crossover at

$$M_{co} = \frac{\sqrt{\mu}}{1+\sqrt{\mu}} < 1, \tag{8}$$

corresponding to

$$\varphi_{co} = \frac{1}{2(1+\sqrt{\mu})^2}. \tag{9}$$

The fact that $\varphi_{\ell c}$ increases and $\varphi_{\ell a}$ decreases as M is increased will be recalled later to explain some of the observed properties of the Sagdeev pseudopotentials. At this stage, it is worth remarking that the limit $M = 1$ is outside the validity of the model, as the nonlinear structure speed is then at the thermal velocity of the adiabatic ions.

Having thus obtained all constituent densities, we introduce their coupling in the normalized form of Poisson's equation,

$$\frac{d^2\varphi}{dx^2} + f n_c + (1-f)n_a - n_e - 0, \tag{10}$$

where the fractional ion densities are $f = \tilde{n}_{c0}/\tilde{n}_{e0}$ and $1 - f = \tilde{n}_{a0}/\tilde{n}_{e0}$, and thus $0 < f \leq 1$. The value $f = 0$ has to be excluded, as there would no longer be a supersonic species and the model describing acoustic modes breaks down. We will later see that $f = 0$ corresponds to $M = 0$, so that this limit on M is also outside the description, even though some of the algebra goes through nicely at the extreme values 0 and 1. At $f = 1$, the description is reduced to the simplest model for ion-acoustic solitons, with cold ions for the inertial and Boltzmann electrons for the pressure effects.

After multiplication by $d\varphi/dx$ and integration, one gets

$$\frac{1}{2}\left(\frac{d\varphi}{dx}\right)^2 + S(\varphi) = 0. \tag{11}$$

This looks like an energy integral in classical mechanics, for a particle with unit mass in a conservative force field, with φ in the role of the particle coordinate and x in the role of time. The potential energy is then $S(\varphi)$, called the Sagdeev potential or pseudopotential,[1]

$$\begin{aligned} S(\varphi, M) = {} & \frac{1}{\tau}(1 - \exp[\tau\varphi]) + f\frac{M^2}{\mu}\left(1 - \sqrt{1 - \frac{2\mu\varphi}{M^2}}\right) \\ & + \frac{1-f}{6}\left\{2 + 6M^2 - [(1+M)^2 - 2\varphi]^{3/2}\right. \\ & \left. - [(1-M)^2 - 2\varphi]^{3/2}\right\}. \end{aligned} \tag{12}$$

The Sagdeev pseudopotentials already satisfy $S(0, M) = 0$ and $S'(0, M) = 0$ in the undisturbed conditions. We denote for brevity derivatives of $S(\varphi, M)$ with respect to φ by primes and omit the explicit dependence on the compositional parameters f, μ, and τ. In order to have the possibility of encountering solitary waves and double layers, the "origin" must be unstable, at least on one side,[2–6] which leads to

$$S''(0, M) = \frac{f\mu}{M^2} - \frac{1-f}{1-M^2} - \tau \leq 0, \tag{13}$$

often called the soliton condition.

We recall that the M value determined from $S''(0, M) = 0$ is precisely the normalized global acoustic speed, M_s, and thus the expression M/M_s represents the true Mach number for the given plasma composition. As we are investigating a model with cold supersonic and adiabatic subsonic ions (for which $M < 1$), the root $M_s < 1$ must be chosen, which yields

$$M_s^2 = \frac{1}{2\tau}\left[\tau + f\mu + 1 - f - \sqrt{(\tau + f\mu + 1 - f)^2 - 4f\mu\tau}\right]. \tag{14}$$

The existence conditions then require that $M_s \leqslant M < 1$.

Recently, we have seen that under certain conditions solitons and/or double layers might exist at the acoustic speed itself, and to investigate such possibilities, one has to explore the properties of $S(\varphi, M_s)$, i.e., the pseudopotential at the acoustic speed.[2–6] Since for this Sagdeev pseudopotential we have $S(0, M_s) = S'(0, M_s) = S''(0, M_s) = 0$, its convexity is determined from the sign of $S'''(0, M_s)$, in our case from

$$S'''(0, M_s) = \frac{3f\mu^2}{M_s^4} - \frac{(1 - f(1 + 3M_s^2)}{(1 - M_s^2)^3} - \tau^2. \tag{15}$$

It is also shown elsewhere[3–6] that for $S'''(0, M_s) > 0$ there exist positive KdV-like solitons, the amplitudes of which vanish when $M \to M_s$. For $S'''(0, M_s) < 0$, the polarities are reversed and we have necessarily negative KdV-like solitons. This is consistent with KdV theory, in which the sign of the soliton is the same as that of the coefficient of the nonlinear term in the KdV equation, and it has been shown that that coefficient is effectively given by $S'''(0, M_s)$ (Ref. 3).

Therefore, interesting phenomena become possible in models for which specific critical parameter values yield $S'''(0, M_s) = 0$, as also noted in some recent papers involving a number of different plasma models.[3–6]

Unfortunately, it becomes impossible to determine analytically whether $S'''(0, M_s) = 0$ might occur or not when (15) is rendered more explicit by substitution of M_s from (14). In Sec. III, we will see that indeed a transition through $S'''(0, M_s) = 0$ is possible for higher values of f, for a wide range of numerical values of μ and τ. However, we shall show that, unlike earlier results,[3–5] for this model the critical condition $S'''(0, M_s) = 0$ is not associated with a coexistence region. Coexistence is used as a shorthand in the literature, with the (implicit) understanding that only one solution can be realized at a time.

III. DISCUSSION

To start the discussion, at small f and M the existence domain of positive solitons is limited by cold ion compression, with $0 < \varphi < \varphi_{\ell c}$, as long as $\varphi_{\ell c} \leqslant \varphi_{co}$; recall the limits defined in (7). As for small $\varphi > 0$ we know that the convexity requires $S(\varphi, M) < 0$, positive solitons will exist if roots can be encountered before the cold ion compression ends the definition range of $S(\varphi, M)$. We will express the condition for the largest possible root before cut-off as $S(\varphi_{\ell c}, M) = 0$ in the limit, which in principle yields a curve *M*(*f*) at given μ and τ. Analytically, as $S(\varphi_{\ell c}, M) = 0$ is linear in f, it is easier to solve this equation for f, yielding

$$f_{\ell c} = \frac{A(M)}{B(M)}, \tag{16}$$

where

$$\begin{aligned} A(M) &= \frac{1}{\tau}(1 - \exp[\tau\varphi_{\ell c}]) + \frac{1}{6}\Big\{2 + 6M^2 \\ &\quad - [(1+M)^2 - 2\varphi_{\ell c}]^{3/2} \\ &\quad - [(1-M)^2 - 2\varphi_{\ell c}]^{3/2}\Big\}, \\ B(M) &= \frac{M^2}{\mu} - \frac{1}{6}\Big\{2 + 6M^2 - [(1+M)^2 - 2\varphi_{\ell c}]^{3/2} \\ &\quad - [(1-M)^2 - 2\varphi_{\ell c}]^{3/2}\Big\}. \end{aligned} \tag{17}$$

After the crossover between cold and warm ion sonic point limitations, occurring at f_{co}, we express the limiting condition as $S(\varphi_{\ell a}, M) = 0$, which in principle yields another curve *M*(*f*) at given μ and τ. We again solve this equation for f, obtaining

$$f_{la} = \frac{C(M)}{D(M)}, \tag{18}$$

where

$$\begin{aligned} C(M) &= \frac{1}{\tau}(1 - \exp[\tau\varphi_{\ell a}]) + \frac{1}{3}(1 + 3M^2 - 4M^{3/2}), \\ D(M) &= \frac{M^2}{\mu}\left(1 - \sqrt{1 - \frac{2\mu\varphi_{\ell a}}{M^2}}\right) - \frac{1}{3}(1 + 3M^2 - 4M^{3/2}). \end{aligned} \tag{19}$$

A note of caution should be sounded at this stage. Conditions like $S(\varphi_{\ell a}, M) = 0$ may lead to a positive root which is apparently the largest possible root before the density cut-off, but which is not accessible from the origin, because there is another root in between. In that case only the first root encountered yields a solitary wave. In case of doubt, the proper Sagdeev pseudopotential curves will elucidate what precisely is going on.

An example of this is illustrated in Fig. 1, which indicates that $\varphi_{\ell a}$ decreases as M is increased, but that the largest

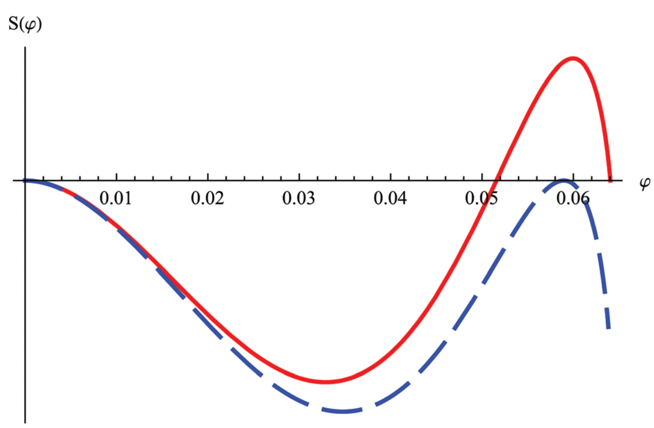

FIG. 1. (Color online) Typical Sagdeev pseudopotentials for $\mu = 1$, $\tau = 1$, and $f = 0.63$. The solid curve is for $M = 0.6421$, such that it ends in a root at $\varphi_{\ell a} = 0.0640$, beyond which it is no longer defined. The dashed curve for $M = 0.6427$ has a positive double layer at $\varphi_{pdl} = 0.0589$ and is no longer defined beyond $\varphi_{\ell a} = 0.0638$.

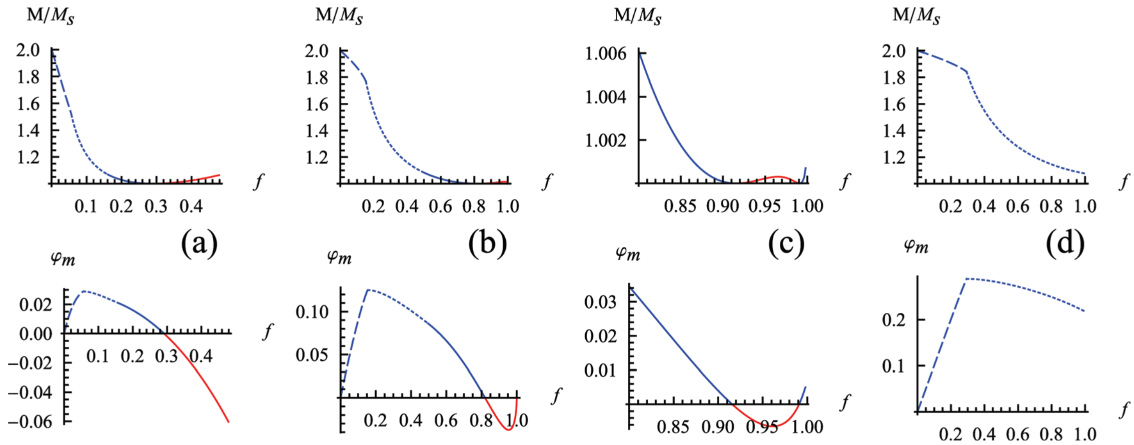

FIG. 2. (Color online) In the upper panels, the existence regions are plotted and in the lower panels, the maximal amplitudes are plotted, for $\tau = 1$. From left to right, the panels are for (a) $\mu = 10$, (b) $\mu = 1$, (c) $\mu = 0.8$, and (d) $\mu = 0.1$. In all panels, dashed curves are at the cold ion infinite compression, dotted curves at the warm ion sonic point, and on the solid curves (when present) double layers occur, first positive, then, for higher f, negative double layers, and possibly a second range of positive double layers. Note that in the panels (c) only the range $0.8 \leqslant f < 1$ has been shown, for graphical clarity, the range for smaller f being qualitatively very similar to the panels (a) and (b). On the other hand, in the panels (d) there are no double layers, as $\mu = 0.1$ is too small to sustain these.

root for the solid curve, although beyond the double layer of the dashed curve, is not accessible. The valid root indicating a soliton is at $\varphi = 0.0516$, and the double layer is actually the upper limit of the sequence of solitons.

If we increase f and M further, another possible limitation of the soliton amplitudes might be reached, if double layers become possible, an example of which has already been shown in Fig. 1. In the model at hand, this new transition (if it exists!) can be determined by requiring that there is a double layer with amplitude $\varphi_{\ell a}$ for some $M = M_{dl}$, i.e., the Sagdeev potential $S(\varphi_{\ell a}, M_{dl})$ yields a double layer. Thus, we determine f and M_{dl} such that $S(\varphi_{\ell a}, M_{dl}) = 0 = S'(\varphi_{\ell a}, M_{dl})$, for given μ and τ, and call the fractional density at which this occurs f_{dl}.

As will be seen from some numerical evaluations, for which double layers can be generated, we always first encounter positive double layers which limit the soliton range, until they go over into negative double layers at the f values given by the critical quadruple root condition $S'''(0, M_s) = 0$. We denote such critical f by f_c, and if more than one can be found in the admissible f range, by f_{c1} and f_{c2}. In the latter case, the double layer polarity changes from positive to negative at f_{c1} and back to positive at f_{c2}. However, as the examples will show, the second transition is usually very close to or even at $f = 1$, so that a limited second range of positive double layers might well be outside the domain of physical acceptability.

Perusal of (12) indicates that $S(\varphi, M)$ is well defined for all negative φ, and goes to $-\infty$ as $\varphi \to -\infty$. This means that if negative roots for $S(\varphi, M)$ occur, they must come in pairs, and then a range of negative solitons is limited by a negative double layer. This is the only possible restriction on the negative φ range, and the absence of negative double layers would prove the absence of negative solitons. However, when accessing the range immediately beyond f_{c1}, where $S'''(0, M_s) < 0$, negative KdV-like solitons necessarily must occur, thus leading to the existence of negative double layers.

In this way, the discussion of possible solitary waves and double layers can be carried out in a systematic fashion and in the order in which the different limitations are encountered:

1. infinite cold ion compression for $0 < f < f_{co}$;
2. warm ion sonic point occurrence for $f_{co} < f < f_{dl}$;
3. positive double layers for $f_{dl} < f < f_{c1}$;
4. negative double layers for $f_{c1} < f < f_{c2}$;
5. positive double layers for $f_{c2} < f < 1$.

In addition, it is important to note that the model requires that $M < 1$, and that constraint may yield a further limit. As we will see, not all ranges exist for all parameter combinations μ and τ.

In what follows, we will pick a number of μ and τ values, and first determine the existence domains in the space of the fractional cold ion density f and the true Mach number (M/M_s) and the maximum solitary wave amplitudes in $[f, \varphi]$ space, as illustrated in Fig. 2 for $\tau = 1$ and values of μ chosen to bring out the generic trends. Figure 2 clearly indicates how for decreasing μ and increasing f the different domains follow sequentially.

For $\mu > 1$, there is only one f_c. Both positive and negative double layers are found, but the negative double layer range is limited by the requirement that $M < 1$, long before $f = 1$ is reached. This case is illustrated in Fig. 2(a) for $\mu = 10$.

For $0.8 \leqslant \mu \leqslant 1$, two values of f_c are found, but the upper one is $f_{c2} = 1$ and the limits of the negative double layers now extend to $f = 1$ and $M = 1$, as shown in Fig. 2(b) for $\mu = 1$. With $f_{c2} = 1$, it follows that the negative double layer amplitude $\to 0$ as $f \to 1$. However, we always have to keep in mind that the limiting cases of $f = 0$ (where $M = 0$) and $f = 1$ are really outside the model considered and that $M < 1$ needs to be obeyed. In particular, $f = 1$ would lead us back to the simplest ion-acoustic model, with Boltzmann electrons and cold ions, where only compressive, positive solitons can be generated. In that case, there is only one mass and one

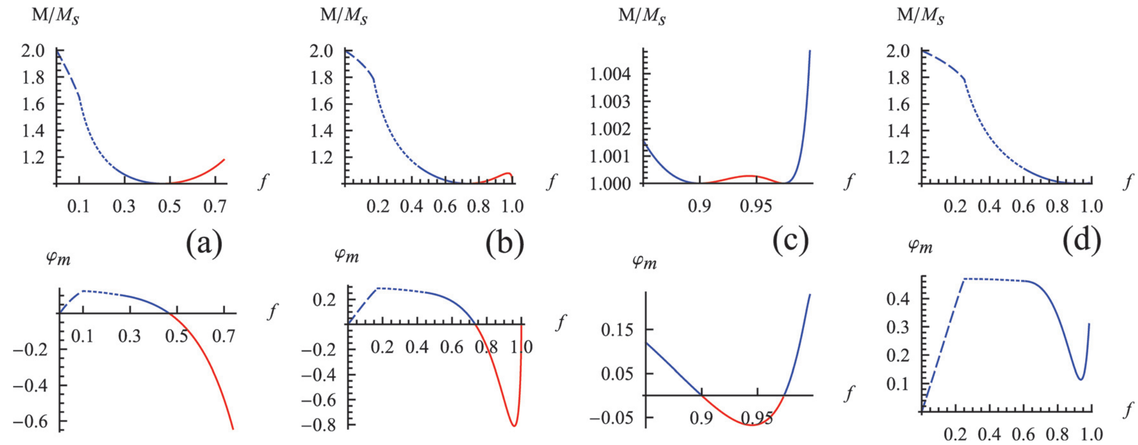

FIG. 3. (Color online) Similar to Fig. 2 but for $\tau = 0.1$, with the same graphical conventions, except that here the panels are for (a) $\mu = 1$, (b) $\mu = 0.1$, (c) $\mu = 0.01$, and (d) $\mu = 0.001$. Again, in panels (c) only the range $0.85 \leqslant f < 1$ has been represented, for graphical clarity.

temperature in the system, and these can be normalized away, so that our results would need to be taken at $\mu = \tau = 1$, but all arguments involving adiabatic ions and double layers do not hold any longer.

In the range $0.7613 \leqslant \mu < 0.8$, there are two effective critical f_c, so that a second set of positive double layers becomes possible beyond the negative double layers, as indicated in Fig. 2(c) for $\mu = 0.8$. This holds until just below $\mu = 0.7613$, when the two f_c coalesce at $f = 0.963$ and then disappear.

Next, for $0.35 \leqslant \mu < 0.7613$ one can no longer satisfy $S'''(0, M_s) = 0$, and there can be no negative double layers, but the positive range survives. This is not illustrated in Fig. 2, but a similar phenomenon will be shown later in Fig. 3(d), for $\mu = 0.001$ and $\tau = 0.1$.

Once $\mu < 0.35$, only the first two limiting conditions survive (infinite cold ion compression and adiabatic sonic point) and we have typical graphs illustrated in Fig. 2(d) for $\mu = 0.1$, up to $f = 1$. Thus only positive solitons and no double layers are found in this range of μ.

An important point to make is that for this model we do not find a coexistence region, in which both positive and negative solitons may occur for the same compositional parameter values and structure speeds. Although in several plasma models,[3–6] the possibility of having $S'''(0, M_s) = 0$ seems to be linked to the occurrence of coexistence ranges in parameter space, the model studied in this paper indeed allows a transition from positive to negative polarity for the KdV-like modes, but no non-KdV-like modes or coexistence ranges have been found, over a wide range of parameter values. Such switches in polarity without coexistence have also been noted previously for other models.[8,44–51] Because in the absence of coexistence all solitary modes are necessarily KdV-like,[3–6] the parameter values for the polarity changes could also have been obtained from a reductive perturbation approach for weakly nonlinear solitary modes but that is outside the scope of the present paper.

Thus, contrary to indications gleaned from recent studies,[3–6] it follows that the condition $S'''(0, M_s) = 0$ is in itself not sufficient to ensure coexistence. For that it is necessary that $S(\varphi, M_s)$ admits a solitary structure (by definition at the acoustic speed) before the model breaks down due to fluid limits.

Much the same conclusions can be drawn from Fig. 3, for $\tau = 0.1$ and various values of μ. We note that for this lower value of τ, the interesting range of μ is similarly reduced, running from 1 to 0.001. When μ is large enough, $\mu > 0.1$, the negative double layer regimes are limited by the requirement that M should not reach 1, occurring before $f = 1$ is reached, as shown in Fig. 3(a) for $\mu = 1$.

As μ is decreased, the regimes are qualitatively much the same, and occur in the same order, now up to $f = 1$, and as soon as $S'''(0, M_s)$ has two effective roots instead of one, a small region near $f = 1$ indicates that a return to positive polarity is theoretically possible. The limiting values are $f_{c2} = 1$ down to $\mu = 0.021$, and then f_{c1} and f_{c2} tend to each other, until they coalesce at $f = 0.942$ for $\mu = 0.0061$. Examples are illustrated in Fig. 3(b) and 3(c), for $\mu = 0.1$ and $\mu = 0.01$, respectively. Once μ drops below 0.0061, $S'''(0, M_s)$ has no roots, but positive double layers can be found for values as low as $\mu = 0.00001$, below which the numerical accuracy can no longer be trusted.

To conclude this part of the discussion, we have carried out similar trials with $\tau = 0.01$ and found essentially the same trends. These are not shown here, as the graphs are qualitatively similar to the ones given already.

To see that our results are not based on numerical artefacts, we have plotted in Fig. 4 the typical Sagdeev pseudopotentials for the combination of $\mu = 1$ and $\tau = 1$, corresponding to the panels in Fig. 2(b). In each panel, the dashed curve is for $S(\varphi, M_s)$, and the solid curve for $S(\varphi, M)$ at the relevant upper limit for M. The dotted curve is for an intermediate value of M.

The values of f have been chosen in such a way that each of the different limitations on the existence of solitary waves is dealt with, as an illustration of the importance of first determining the proper parameter ranges before drawing Sagdeev pseudopotentials. The dashed curves referring to the pseudopotential at the acoustic speed, $S(\varphi, M_s)$, are included to show that the sign of $S'''(0, M_s)$ determines the

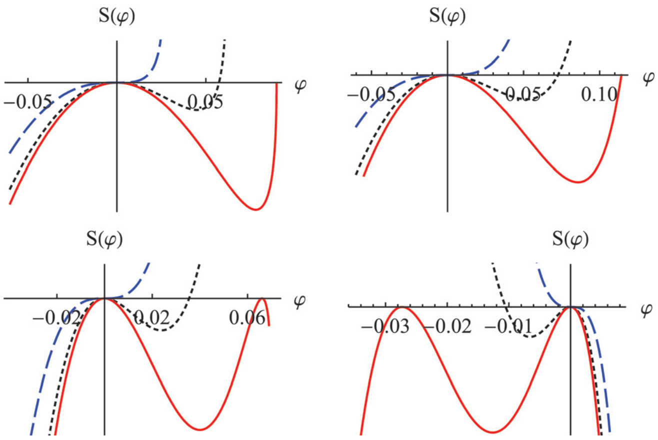

FIG. 4. (Color online) Typical Sagdeev pseudopotentials for $\mu = 1$ and $\tau = 1$. In the four panels, the solid curves are for M corresponding to the relevant limitation, the dashed curves represent $S(\varphi, M_s)$, and the dotted curves are for an intermediate M. Left upper panel: solitons limited by cold ion compression, for $f = 0.1$, with $M_s = 0.227$ and $M_{\ell c} = 0.424$; right upper panel: solitons limited by adiabatic ion sonic point, for $f = 0.3$, with $M_s = 0.404$ and $M_{\ell a} = 0.424$; left lower panel: solitons limited by positive double layers, for $f = 0.6$, with $M_s = 0.606$ and $M_{pdl} = 0.628$; right lower panel: solitons limited by negative double layers, for $f = 0.9$, with $M_s = 0.827$ and $M_{ndl} = 0.831$.

polarity of the (KdV-like) solutions. As we have not found any coexistence regions, no non-KdV-like solitary waves are supported by our plasma model.

It is clear that we could illustrate the other combinations of μ and τ shown in Figs. 2 and 3, but the trends are very similar and such additional figures would not offer really new insight. We will merely pick a small number of examples to add some additional clarifications.

First, in Fig. 5, we show Sagdeev pseudopotentials for $\mu = 1$, $\tau = 0.1$, and $f = 0.7$, numbers which are in the realm of Fig. 3(a), where the existence range in f is limited by not breaking through the $M = 1$ barrier. The solid curve shows a negative double layer solution at $\varphi_m = -0.479$ for $M = 0.928$, whereas the dashed curve is at the acoustic speed $M_s = 0.824$, indicating that $S'''(0, M_s) < 0$. Note that both Sagdeev pseudopotentials exist for all negative φ and ultimately go down to $-\infty$, so that only part of the range is shown. To the contrary, the positive φ range has been drawn in full but is very restricted, as the corresponding adiabatic sonic point limitations are so small, at $\varphi_{\ell a} = 0.0156$ (solid curve) and $\varphi_{\ell a} = 0.0026$ (dashed curve), respectively, beyond which the Sagdeev pseudopotentials do not exist any longer. Hence, no well is formed on that side.

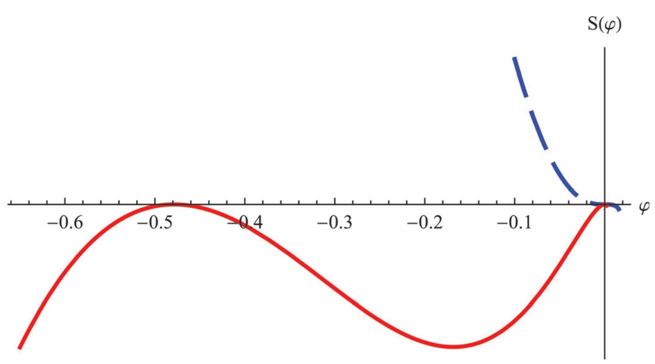

FIG. 5. (Color online) Sagdeev pseudopotentials for $\mu = 1$, $\tau = 0.1$, and $f = 0.7$, where the solid curve has a negative double layer solution at $\varphi_m = -0.479$ for $M = 0.928$, whereas the dashed curve is at the acoustic speed $M_s = 0.824$ and has no soliton solution.

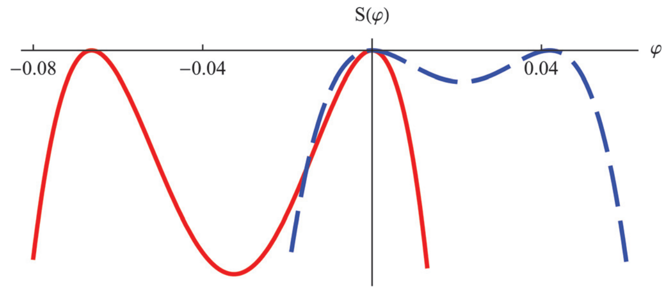

FIG. 6. (Color online) Two Sagdeev pseudopotentials for $\mu = 0.01$ and $\tau = 0.1$, at different f, the solid curve having a negative double layer at $\varphi_m = -0.0662$ for $f = 0.95$ and the dashed curve a positive double layer at $\varphi_m = 0.0419$ for $f = 0.98$.

Next, we illustrate in Fig. 6 two Sagdeev pseudopotentials, one having a negative double layer at $\varphi_m = -0.0662$ for $f = 0.95$ and $M = 0.249$ (solid curve), the other a positive double layer at $\varphi_m = 0.0419$ for $f = 0.98$ and $M = 0.284$ (dashed curve). These conditions correspond to the panels in Fig. 3(c), for two values of f that are on either side of $f_{c2} = 0.974$. Because these Sagdeev potentials are drawn for different f and M, their curves can cross. As the existence diagram in Fig. 3(c) indicates, the true Mach numbers at which the double layers occur are barely above 1, and the double layers themselves are hence of small amplitude.

Finally, we plot in Fig. 7 two examples of Sagdeev pseudopotentials corresponding to the conditions of Fig. 3(d). The solid curve is for $f = 0.8$ and $M = 0.0524$, with a positive double layer at $\varphi = 0.338$, whereas the dashed curve is for $f = 0.9$ and $M = 0.0671$, with a positive double layer at $\varphi = 0.153$. Because of the shallowness of the Sagdeev pseudopotential for the latter, between the double roots at 0 and at 0.153, the double layer is not easily discernible, even though it is not exactly small. The very shallow well implies that the slope of the electrostatic double layer profile is very gentle, indeed.

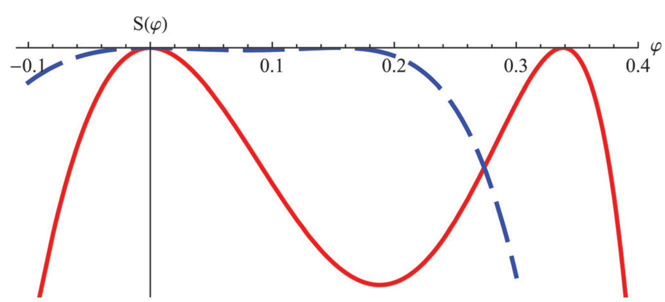

FIG. 7. (Color online) Sagdeev pseudopotentials for $\mu = 0.001$ and $\tau = 0.1$. The solid curve is for $f = 0.8$ and has a positive double layer at $\varphi = 0.338$, whereas the dashed curve is for $f = 0.9$, with a positive double layer at $\varphi = 0.153$.

IV. CONCLUSIONS

We have investigated the occurrence of large nonlinear acoustic waves in a plasma made up of cold supersonic and adiabatic subsonic positive ions, in the presence of hot isothermal electrons, with the help of the appropriate Sagdeev pseudopotential theory. In this model, there are no solitons at the acoustic speed, and indeed, we have not found compositional parameter ranges where solutions of opposite polarities can coexist. All nonlinear modes are thus super-acoustic and KdV-like, requiring $M > M_s$, but the upper limits on admissible M come from different physical arguments.

First, there are limitations on positive solitons coming from the ion densities, in that the cold supersonic ions reach an infinite density compression, or the adiabatic subsonic ions encounter their sonic point, both limiting the positive polarity modes. Another limitation occurs when double layers are encountered, and here, for a number of combinations of the mass ratio $\mu = m_a/m_c$ and the temperature ratio $\tau = T_a/T_e$, solitary wave polarities can switch from positive to negative, and even sometimes back to positive, depending on possible changes of sign of $S'''(0, M_s)$, as the fractional cold ion density $f = \tilde{n}_{c0}/\tilde{n}_{e0}$ is increased. In this way, the discussion of possible solitary waves and double layers has been carried out in a systematic fashion, and it is found that the different limitations are encountered in a definite order when f is increased: infinite cold ion compression, warm ion sonic point, positive double layers, negative double layers, and finally positive double layers.

However, not all possible ranges exist for all parameter combinations μ and τ. Whereas the cold and warm ion sonic point limitations seem to be present in all cases, and thus positive polarity solutions can easily be obtained, double layers have a more restricted existence range, usually toward higher f values, and requiring at least intermediate to high values of μ at given τ, particularly, if polarity changes are sought. Such conclusions are similar to those obtained for other three- or four-component plasma models[8,46,47] but have now been arrived at in a much more systematic way, by studying the special Sagdeev pseudopotential at the acoustic speed, $S(\varphi, M_s)$, and finding critical parameter values for which $S'''(0, M_s)$ can change sign. This is indeed linked to a switch in the polarity of the nonlinear acoustic structure.

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