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Electrostatic solitary waves in the presence of excess superthermal electrons: modulational instability and envelope soliton modes

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Abstract
The nonlinear dynamics of electrostatic solitary waves in the form of localized modulated wavepackets is investigated from first principles. Electron-acoustic (EA) excitations are considered in a two-electron plasma, via a fluid formulation. The plasma, assumed to be collisionless and uniform (unmagnetized), is composed of two types of electrons (inertial cold electrons and inertialess kappa-distributed superthermal electrons) and stationary ions. By making use of a multiscale perturbation technique, a nonlinear Schrödinger equation is derived for the modulated envelope, relying on which the occurrence of modulational instability (MI) is investigated in detail. Stationary profile localized EA excitations may exist, in the form of bright solitons (envelope pulses) or dark envelopes (voids). The presence of superthermal electrons modifies the conditions for MI to occur, as well as the associated threshold and growth rate. The concentration of superthermal electrons (i.e., the deviation from a Maxwellian electron distribution) may control or even suppress MI. Furthermore, superthermality affects the characteristics of solitary envelope structures, both qualitatively (supporting one or the other type, for different $\kappa$) and quantitatively, changing their characteristics (width, amplitude). The stability of bright and dark-type nonlinear structures is confirmed by numerical simulations.

(Some figures in this article are in colour only in the electronic version)

1. Introduction
Electron-acoustic (EA) waves (EAWs) occur in plasmas which are characterized by coexisting electron populations at different temperatures (to be typically referred to as ‘cold’ and ‘hot’
electrons) [1–6]. These are high-frequency (though below the electron plasma frequency) electrostatic electron oscillations, sustained thanks to the pressure of the hot electron component, acting as the restoring force, while the cold electron component provides the inertia [1, 7]; the ions may be thought of as plainly providing a neutralizing background.

The phase speed $v_{ph}$ of EA waves is much larger than the thermal speeds $v_{ph,c}$ and $v_{ph,i}$ of both cold electrons and ions, yet much smaller than the thermal speed $v_{ph,h}$ of the hot electron component. The frequency of EAWs is typically well below the cold electron plasma frequency, since the wavelength is larger than the Debye length $\lambda_{D,h} = (T_h/4\pi n_{h0}e^2)^{1/2}$ involving the temperature $T_h$ and the number density $n_{h0}$ of the hot electron component. Despite one’s intuitive expectation, EAWs survive Landau damping in a wide range of parameters, in fact approximately in the region $T_h/T_c \geq 10$ and $0.25 \leq n_c/n_{e,tot} = \beta \leq 4$ (i.e., $0.2 \leq n_c/n_{e,tot} \leq 0.8$ and $n_{e,tot} = n_c + n_h$) [3, 7], expressed in terms of the temperature ($T_c, T_h$) and density ($n_c, n_h$) of the electron constituents (indices ‘c’ for cold, ‘h’ for hot). Furthermore, a study of the plasma dispersion function from first principles suggests that Landau damping is minimized in a wide range of wavenumber $k$ values for every value of $\kappa$ [8].

Electrostatic solitary wavepackets (ESW) are observed in abundance in space plasmas, also reported in satellite recorded data, e.g., in the auroral region [9–11]. Such observed events have tacitly been attributed to EAWs. Theoretical studies of EAWs in space plasmas [12–14] have been complemented by experimental observations in pure ion plasmas [15], in pure electron plasmas [16] and in laser produced plasmas [17].

Electron distribution measurements in near-Earth space environments suggest a (often strong) deviation from Maxwellian equilibrium [18–20], due to the presence of accelerated, energetic (superthermal) particles, which may arise due to the effect of external forces or to wave particle interaction. A similar situation has been observed experimentally [21]. Such deviations from the Maxwellian distribution are expected to exist in any low density plasma or in the interstellar medium, where binary collisions are sufficiently rare. The effect of a nonthermal electronic distribution has been studied via the original Cairns ansatz for ion-acoustic solitary structures [22], to interpret the data observed by the FREJA satellite; the same method was later applied to EA solitary waves [23]. An alternative approach to nonthermality consists in adopting the so-called kappa distribution function [24–26], originally employed in [24] to explain the power-law dependence in velocity space with high-energy tails [27] observed in space; this distribution function was named after the parameter $\kappa$ (kappa), which determines the high-energy power-law index. The kappa-function resembles a Maxwellian distribution at low energies, making a smooth transition into a power-law tail at much higher energies. If the spectral index $\kappa$ becomes very large (i.e., when $\kappa \to \infty$) a Maxwellian distribution is recovered [28–30]. The kappa distribution has now been used to model different plasma situations [31–35], and has been argued to model space plasmas more accurately by the kappa distribution than by a superposition of Maxwellian distributions [24, 29, 36]. The presence of a high-energy tail component in a kappa distribution considerably changes the rate of resonant energy transfer between particles and plasma waves, so that the conditions for Landau damping and plasma instabilities may be substantially different for the two distributions.

A well known nonlinear mechanism involved in plasma wave dynamics is amplitude modulation, which may be due to parametric wave coupling, interaction between high and low frequency modes or simply to the nonlinear self-interaction of the carrier wave. We focus particularly on modulational instability (MI), a nonlinear mechanism associated with harmonic generation in plasmas. The standard method to study this mechanism adopts a multiple (space and time) scales technique [37, 38], which generally leads to a nonlinear Schrödinger equation (NLSE) describing the evolution of a slowly varying wave wavepacket envelope. Under certain conditions, waves may develop a Benjamin–Feir-type (modulational) instability, i.e.,
their modulated envelope may either become unstable to a small external perturbation or it may evolve into a train of solitary waves (solitons). The final stage represents wave energy localization and re-distribution into localized envelope modes [39]. As regards electrostatic plasma waves described via a fluid model, the method has been discussed thoroughly in [40], while it has also been employed for modulated EAWs (against a Maxwellian background) [41], a standard reference in what follows.

Our aim here is to investigate the nonlinear dynamics of modulated EA wavepackets. We consider an unmagnetized plasma composed of stationary ions, inertial cold electrons and inertialess, kappa-distributed hot electrons. We examine the occurrence of MI as well as the existence of envelope solitary structure associated with EAWs. This paper is organized in the following manner. In section 2, we present the relevant fluid equations for EAWs. Adopting a multiple scales perturbation method, we discuss the linear behavior of the EAWs and then derive the NLSE governing the dynamics of modulated EAWs in section 3. A nonlinear dispersion relation is derived in section 4 for the amplitude modulation, and the occurrence of MI is pointed out. Section 5 is dedicated to a summary of envelope soliton solutions of the NLSE. A parametric investigation is presented in section 6, in terms of the various parameters involved (carrier wavenumber and superthermal plasma composition), and the resulting predictions are tested by a series of numerical simulations. We summarize our results in section 7.

2. Electron fluid model

We consider a three-component collisionless unmagnetized plasma, consisting of ‘cold’ inertial electrons (charge $-e$ and mass $m_e$), kappa-distributed inertialess ‘hot’ electrons and immobile ions (charge $q_i = +Ze$ and mass $m_i$; the charge state $Z_i$ is left arbitrary. We ignore the inertia of the hot electron component by assuming that the characteristic hot electron (‘thermal’) speed is much higher than the wave phase speed, a fact which is also a requirement for Landau damping to be minimized [32]. The number density $n_c$ and the velocity $u_c$ of the cold electron fluid is governed by the continuity and momentum equations

$$\frac{\partial n_c}{\partial t} + \nabla \cdot (n_cu_c) = 0,$$

$$\frac{\partial u_c}{\partial t} + (u_c \cdot \nabla)u_c = \frac{e}{m_e}\nabla \Phi,$$

while the electric potential $\Phi$ is obtained from Poisson’s equation:

$$\nabla^2 \Phi = 4\pi e (n_c + n_h - Z_in_i).$$

In the latter relation, the ion density is assumed to be constant ($n_i = n_{i0} = \text{const.}$), while a kappa distribution is adopted for the hot electrons [24–26]

$$n_h = n_{h0} \left[ 1 - \frac{e\Phi}{(\kappa - \frac{e}{2}k_BT_h)} \right]^{-\kappa+1/2},$$

where the real parameter $\kappa$ measures the deviation from Maxwellian equilibrium (which is recovered in the limit $\kappa \to \infty$ at every step; henceforth to be referred to as the Maxwellian limit). Note that $\kappa > 3/2$ for physically meaningful distributions [26]. Charge neutrality at equilibrium implies

$$Z_in_{i0} - n_{c0} - n_{h0} = 0,$$

where the index ‘0’ denotes the unperturbed (equilibrium) number-density values.
It is convenient at this stage to scale all variables by appropriate quantities, to obtain a dimensionless system of equations. The normalized one-dimensional continuity, momentum and Poisson’s equation take the form

\[
\frac{\partial n}{\partial t} + u \frac{\partial (n u)}{\partial x} = 0, \\
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \frac{\partial \phi}{\partial x}, \\
\frac{\partial^2 \phi}{\partial x^2} \approx \beta(n - 1) + c_1 \phi + c_2 \phi^2 + c_3 \phi^3,
\]

(6) (7) (8)

where \( u_c, n_c, \) and \( \Phi \) are scaled as \( u = u_c/v_0, \) \( n = n_c/n_{00} \) and \( \phi = \Phi/\Phi_0. \) Space and time variables are scaled by the Debye length \( \lambda_{\text{D},h} = (k_B T_h/4\pi n_{00} e^2)^{1/2} \) and the inverse hot electron plasma frequency \( \omega_{\text{peh}} = (4\pi n_{00} e^2/m_e)^{1/2}, \) respectively, while the potential scale reads \( \Phi_0 = k_B T_h/e. \) The characteristic speed scale adopted is \( v_0 \equiv (k_B T_h/m_e)^{1/2}. \) We note that we have expanded the right-hand side (rhs) of Poisson’s equation near equilibrium. The dimensionless parameters appearing in equation (8) are the cold-to-hot electron component density ratio \( \beta \)

\[
\beta = n_{c0}/n_{h0},
\]

(9)

and the \( \kappa \)-related coefficients

\[
c_1 = \frac{\kappa - 1/2}{\kappa - 3/2}, \\
c_2 = \frac{(\kappa - 1/2)(\kappa + 1/2)}{2(\kappa - 3/2)^2}, \\
c_3 = \frac{(\kappa - 1/2)(\kappa + 1/2)(\kappa + 3/2)}{6(\kappa - 3/2)^3},
\]

(10)

in fact all positive quantities. It should be noted, for rigor, that the convergence of the series appearing in rhs (8) implies \( \kappa \geq 3, \) practically, for the expansion to be valid. It is interesting to point out that the ion charge state \( Z_i \) has been scaled out in the system (6)–(8) due to normalization, since we have set \( Z_i n_{i0} = \beta + 1 \) following equation (5). Therefore our results are valid for any value of \( Z_i, \) that is, either for hydrogen \( (Z_i = 1) \) or heavier \( (Z_i > 1) \) ions.

3. Amplitude modulation: perturbative analysis and results

The dynamics of a slowly varying amplitude (envelope) can be studied via a multiple scales perturbation technique [37, 38], involving an expansion near equilibrium in terms of an \( \textit{ad hoc} \) smallness parameter \( \epsilon \ll 1. \) The (fast) carrier wave and the (slow) envelope are thus assumed to evolve in order(s) \( \sim \epsilon^1 \) and \( \sim \epsilon^2 \) (or higher), respectively. Details on the method can be found elsewhere (see, e.g., [40]), so only the basic steps are provided here.

Let \( S \) be the state (column) vector \( (n, u, \phi)^T, \) describing the system’s state at a given position \( x \) and instant \( t. \) Small deviations will be considered from the equilibrium state \( S^{(0)} = (1, 0, 0)^T \) by taking \( S = S^{(0)} + \epsilon S^{(1)} + \epsilon^2 S^{(2)} + \cdots = S^{(0)} + \sum_{n=1}^{\infty} \epsilon^n S^{(n)}, \) where \( \epsilon \ll 1 \) is a small (real) parameter. The wave amplitude is thus allowed to depend on the stretched (slow) coordinates \( X_n = e^{\epsilon x} \) and \( T_n = e^{\epsilon t}, \) where \( n = 1, 2, 3, \ldots \) (namely \( X_1 = \epsilon x, \) \( X_2 = \epsilon^2 x \) and so forth; same for time), distinguished from the (fast) carrier variables \( x (\equiv X_0) \) and \( t (\equiv T_0). \)

All the perturbed states depend on the fast scales via the phase \( \theta_1 = kx - \omega t \) only, while the slow scales only enter the \( l \)th harmonic amplitude \( S^{(n)}_{l}. \) Secondary harmonic is accounted for via the ansatz \( S^0 = \sum_{l=-\infty}^{\infty} S_{l}^{(0)}(X, T) e^{\imath(lkx-\omega t)}; \) the reality condition \( S_{-l}^{(n)} = S_{l}^{(n)*} \) is met by all state variables.
Figure 1. Variation of the electrostatic screening (effective Debye) length (scaled) with the superthermality parameter $\kappa$. See that the curve approaches unity at the infinite $\kappa$ limit.

**Linear limit.** Substituting into equations (6)–(8) and isolating the contributions for $n = l = 1$, we obtain the linear system

\begin{align*}
-i\omega n^{(1)}_1 + ik u^{(1)}_1 &= 0, \\
-i\omega u^{(1)}_1 - ik \phi^{(1)}_1 &= 0, \\
-\beta n^{(1)}_1 - (k^2 + c_1) \phi^{(1)}_1 &= 0,
\end{align*}

whose solution provides the first harmonic amplitudes. The compatibility (solvability) condition leads to the dispersion relation

$$\omega^2 = \frac{k^2 \beta}{k^2 + c_1}. \quad (14)$$

For large wavelengths (i.e., for $k \ll \sqrt{c_1}$) the EA wave frequency reads

$$\omega \approx k \sqrt{\frac{\beta}{c_1}} \quad (15)$$

which clearly shows the dependence of the phase velocity ($v_{ph} = \omega/k$) of EAWs on the superthermality parameter (via $\kappa$) and on the cold electron concentration as well (via $\beta$). See that stability is ensured by the (obvious, by construction) positivity of $c_1$. It is easily seen that $v_{ph}$ decreases with increasing superthermality (decreasing $\kappa$), yet increases with an increasing cold electron density (higher $\beta$). Restoring dimensions in equation (14), one can easily see that the effective ($\kappa$-dependent) screening length $\lambda_{D, eff}^{(k)} = [(\kappa - 3/2)/(\kappa - 1/2)]^{1/2} \lambda_{D,h}$ increases with $\kappa$, as shown in figure 1, while it is larger for a plasma containing less energetic particles. This is an agreement with [31, 32].

From equations (11)–(13) we can also determine the first harmonics of the perturbation as

$$n^{(1)}_1 = -\frac{\omega^2}{k^2} \phi^{(1)}_1, \quad u^{(1)}_1 = -\frac{k}{\omega} \phi^{(1)}_1. \quad (16)$$

**Second order in $\epsilon$.** Applying the same procedure for $n = 2$, we obtain the amplitude(s) of the second harmonics $S_{2}^{(2)}$ and the constant (direct current) terms $S_{0}^{(2)}$, as well as a finite contribution $S_{1}^{(2)}$ to the first harmonics. All of these quantities can be expressed, e.g., in terms of the first order potential correction $\phi^{(1)}_1$; the detailed expressions are provided in the appendix. The equations for $n = 2, l = 1$ provide the compatibility condition:

$$\frac{\partial \phi^{(1)}_1}{\partial T_1} + v_T \frac{\partial \phi^{(1)}_1}{\partial X_1} = 0, \quad (17)$$

where
where the group velocity \( v_g \) is defined as

\[
v_g = \frac{d\omega}{dk} = \frac{\omega}{k^3} c_1 \beta.
\] (18)

This constraint, readily verified by imposing that \( \phi^{(1)}_1 \) (and consequently all first order harmonic amplitudes) should be functions of the moving coordinate \( \xi = X_1 - v_g T_1 \), represents an envelope moving at the group velocity, to this order.

Nonlinear Schrödinger amplitude evolution equation. Proceeding to the third order in \( \epsilon \) (\( n = 3 \)), the equations for \( l = 1 \) yield an explicit compatibility condition in the form of the NLSE

\[
i \frac{\partial \psi}{\partial \tau} + P \frac{\partial^2 \psi}{\partial \xi^2} + Q|\psi|^2 \psi = 0,
\] (19)

where \( \psi \) denotes the electric potential correction \( \phi^{(1)}_1 \), for simplicity, and the slow variables are \( \xi = X_1 - v_g T_1 \) and \( \tau = T_2 \).

The group dispersion coefficient \( P \) is related to the curvature of the dispersion curve as

\[
P = \frac{1}{2} \frac{d^2 \omega}{dk^2}.
\]

The exact form of \( P \) reads

\[
P(k; \kappa, \beta) = -\frac{3}{2} \frac{\omega^3}{c_1} \frac{1}{k^4} \beta^2,
\] (20)

which is a negative quantity. The nonlinearity coefficient \( Q \) is due to the carrier wave self-interaction in the background plasma and can be written in the form

\[
Q(k; \kappa, \beta) = \frac{\omega^3}{\beta k^2} \left[ c_2 \left( C_3^{(22)} + C_3^{(20)} \right) + \frac{3}{2} c_3 \right] - k \left[ C_2^{(22)} + C_2^{(20)} \right] - \frac{\omega}{2} \left( v_2^{(22)} + v_2^{(20)} \right).
\] (21)

All coefficients (e.g., \( C_2^{(22)} \), \( C_2^{(20)} \), ...) appearing in equation (21) can be found in the appendix.

Let us examine the long-wavelength behavior (i.e., for \( k \ll 1 \)) of the dispersion coefficient \( P \) and the nonlinear coefficient \( Q \). We find that \( P \) varies linearly with \( k \):

\[
P \approx -\frac{3}{2} \frac{\omega^3}{k^3} \left( \frac{2 \kappa - 3}{2 \kappa - 1} \right)^{3/2} \beta^{1/2} k < 0,
\] (22)

while on the other hand the nonlinear coefficient \( Q \) varies as \( \sim 1/k \) for small wavenumber \( k \):

\[
Q \approx \frac{2(\beta + 3)\kappa + \beta - 3}{12 \beta^{3/2}} \left( \frac{2 \kappa - 1}{(2 \kappa - 3)^{5/2}} \right) k > 0.
\] (23)

It is straightforward to see that the dispersion coefficient \( P \) is always negative for any wavenumber (cf figure 2), while the nonlinearity coefficient \( Q \) is negative for shorter wavelengths. In any case, in the long-wavelength limit we have \( P < 0 < Q \), ensuring modulational stability for large wavelengths (refer to the discussion below).

For the sake of reference, let us consider the infinite \( \kappa \) limit, in which we obtain \( P \approx -\frac{3}{2} k \sqrt{\beta} \) and \( Q \approx (1/k)((\beta + 3)^2/12 \beta^{3/2}) > 0 \); these exactly match the expressions in [41]. Interestingly, for finite \( \kappa \), and for given \( k \), say, \( P \) decreases in absolute value, while \( Q \) increases (for lower \( \kappa \) that is), compared with the Maxwellian case (also see figure 2). The ratio \( P/Q \) therefore acquires smaller values as \( \kappa \) gets smaller, suggesting that excess superthermality reduces the width of the envelope solitons; see equation (31).

In a general manner, the evolution of a wavepacket whose amplitude obeys the NLSE equation (19) essentially depends on the dispersion and nonlinearity coefficients, \( P \) and \( Q \). In
this study, our aim is to analyse the effect of superthermality (via $\kappa$) and plasma concentration (via the cold-to-hot electron density ratio $\beta$) on these two coefficients. Standard results can be found in the literature [39, 40], so only the essential analytical steps will be summarized in the following two sections, devoted to MI and to localized excitations (envelope solitons).

4. Modulational stability analysis

The standard linear (modulational) stability analysis consists in considering the solution of equation (19), $\psi = \psi_m e^{i\theta}$, where a monochromatic solution is $\tilde{\psi} = \psi_0 e^{i(\tilde{k} \tilde{\zeta} - \tilde{\omega} \tau)}$. We consider the linear perturbation by setting $\psi_m = \psi_0 + \epsilon \tilde{\psi}_1$, where $\tilde{\psi}_1 = \tilde{\psi}_{1,0} e^{i(\tilde{k} \tilde{\zeta} - \tilde{\omega} \tau)}$ (the perturbation wavenumber $\tilde{k}$ and the frequency $\tilde{\omega}$ are to be distinguished from their carrier wave homologue quantities, denoted by $k$ and $\omega$). Substituting this into equation (19), one readily obtains the nonlinear dispersion relation

$$\tilde{\omega}^2 = P^2 \tilde{k}^2 \left( \tilde{k}^2 - 2 \frac{Q}{P} |\psi_0|^2 \right).$$

(24)

If the ratio $P/Q$ is negative, i.e., $P/Q < 0$, the wave will be modulationally stable. If $P/Q > 0$, on the other hand, $\tilde{\omega}^2$ becomes negative for values of $\tilde{k}$ below

$$\tilde{k}_{cr} = \left( 2 \frac{Q}{P} \right)^{1/2} |\psi_0|$$

(25)

that is, of wavelength above a threshold $\lambda_{cr} = 2\pi/\tilde{k}_{cr}$. A purely growing mode then develops. Defining the instability growth rate $\sigma = |\text{Im}\tilde{\omega}(\tilde{k})|$, we find that it reaches its maximum value for $\tilde{k} = \tilde{k}_{cr}/\sqrt{2} = (Q/P)^{1/2} |\psi_0|$, namely

$$\sigma_{\text{max}} = |\text{Im}\tilde{\omega}|_{\tilde{k}=\tilde{k}_{cr}/\sqrt{2}} = |Q||\psi_0|^2.$$

(26)

The dependence of the instability growth rate on $\kappa$ can be investigated using equation (24). Rescaling, we obtain the following expression for the growth rate (note the $\kappa$ dependence, indicated as subscript):

$$\Gamma_{\kappa} = p x \left( 2 \frac{q}{p} - x^2 \right)^{1/2},$$

(27)
Figure 3. Variation of the growth rate $\Gamma_\kappa$ as given by equation (27) with the perturbation wavenumber $x$ for different values of $\kappa$. Here $\beta = 0.5$ for $k = 3.4$.

Figure 4. Variation of the growth rate $\Gamma_\kappa$ as given by equation (27) with the perturbation wavenumber $x$ for different values of $\beta$. Here $\kappa = 4$ and $k = 3.7$.

where $\Gamma_\kappa = \tilde{\omega}/(Q_\infty |\psi_0|^2)$, $p = P/P_\infty$, $q = Q/Q_\infty$ and $x = \tilde{k}/(Q_\infty/P_\infty)^{1/2}|\psi_0|$ with $P_\infty = P(\kappa \to \infty)$ and $Q_\infty = Q(\kappa \to \infty)$. Equation (27) should be compared with the modulational growth rate for the Maxwellian case ($\kappa \to \infty$):

$$\Gamma_\infty = x(2 - x^2)^{1/2},$$

which represents a maximum at $(x_0, \Gamma_{\infty,\text{max}}) = (1, 1)$, and a root at $(\sqrt{2}, 0)$. The effect of excess superthermality (via $\kappa$) on the MI growth rate is depicted in figure 3 (considering a representative wavenumber in the unstable region). It is seen that the MI growth rate is significantly affected by the value of $\kappa$: in fact, the MI growth rate appears to decrease with increasing superthermality, i.e. the instability may be suppressed by excess superthermality (i.e. for lower values of $\kappa$).

Excess superthermal electrons also affect the MI growth rate through the plasma concentration, via the percentage of superthermal electrons (expressed by the ratio $\beta$). This effect is depicted in figure 4 (choosing a representative carrier wavenumber $k$ allowing for instability to occur). The MI growth rate appears to increase for decreasing values of $\beta$, suggesting that the instability is enhanced (i.e., growth rate increased) by a higher concentration of superthermal electrons (namely, lower $\beta$). This qualitative effect appears somehow to contradict the conclusion of the previous paragraph (i.e., instability suppressed by lower $\kappa$, namely stronger superthermality). We see that nonthermality affects the envelope stability in various ways, and the
dynamics is richer than what may simply be shown via a parametrized (in $\kappa$, $\beta$) fluid model.

Concluding this section, we retain that the MI condition depends only on the sign of the $P/Q$ ratio, which may be studied numerically, relying on the exact expressions derived above. This ratio also determines the instability threshold $\tilde{k}_{cr}$ (see above). Once the condition for MI is ensured, the growth rate is given by equation (27). A parametric study is presented below.

5. Envelope solitons

The NLSE (19) is a widely studied partial-differential equation (PDE), which (among other remarkable properties, including integrability [39, 42–44]) possesses exact localized solutions in the form of envelope solitons [45–48]. These are found by seeking a solution in the form $\psi(\zeta, \tau) = \rho(\zeta, \tau)e^{i\Theta(\zeta, \tau)}$, where $\rho = |\psi|$ represents the envelope amplitude and $\Theta$ is the nonlinear phase shift due to self-interaction (both vary weakly in space/time). Different types of envelope solitons are thus found, depending on the sign of the dispersion and nonlinearity coefficients ($P$ and $Q$); an exhaustive discussion can be found in [46–48]. The necessary information for our purposes is summarized below.

For $P$ and $Q$ of the opposite sign (namely for $PQ < 0$), i.e., in our model essentially for large wavelengths (or small wavenumbers, in the modulationally stable region; see figure 2), ‘dark’-type solitons exist, i.e., propagating localized envelope voids (holes). Avoiding algebraic details, one general analytical form for dark envelope solitons reads [41]

$$|\psi| = \psi_0 \left[1 - d^2 \text{sech}^2 \left(\frac{\zeta - V\tau}{L}\right)\right]^{1/2}, \quad \Theta = \frac{1}{2P} \left[V\zeta - \left(\frac{V^2}{2} - 2PQ\psi_0\right)\tau\right]$$

(29)

where $\psi_0$ is the asymptotic value of the electric potential amplitude at infinity, $V$ is the propagation speed (a constant) and $L$ is the soliton width, while the positive constant $d$ regulates the depth of the void ($d = 1$ for black solitons or $d < 1$ for gray ones—cf figures 5(a) and (b)). An interested reader is referred to [40, 46–48] for details. Such excitations represent an electric potential envelope hole and may bear either a finite or a vanishing potential value in the origin, and are thus termed as gray or black solitons, respectively [40]. Their form is depicted in figure 5. Since dark solitons exist in the region where MI is excluded (for $PQ < 0$; see previous section), one expects propagating dark envelopes to be a ‘standard’ electrostatic wavepacket form occurring in plasmas, under the (weak or strong, yet anyhow inevitable) effect of self-modulation.

On the other hand, for $P$ and $Q$ of the same sign (for $PQ > 0$), i.e., for shorter wavelengths (or larger wavenumbers; see figure 2), ‘bright’-type solitons exist, i.e., localized envelope pulses of the form depicted in figure 6. The general analytical form of bright solitons reads [40, 41, 46–48]

$$|\psi| = \psi_0 \text{sech} \left(\frac{\zeta - V\tau}{L}\right), \quad \Theta = \frac{1}{2P} \left[V\zeta - \left(\frac{V^2}{2} + \Omega\right)\tau\right]$$

(30)

where $\Omega$ is the oscillating frequency of the bell-shaped envelope solitons for $V = 0$. These bell-shaped envelope excitations are qualitatively analogous to bright pulses in nonlinear optics [49] (e.g., in optical fibers), which is not surprising, given the analogous physics (dispersion management via nonlinearity) involved.

Interestingly, in both of the latter two expressions, the constants $\psi_0$ and $L$ are related by

$$L = \frac{1}{\psi_0} \left(\frac{2P}{|Q|}\right)^{1/2},$$

(31)
suggesting a straightforward way to diagnose and identify such solitons in observations, and also to predict their geometric characteristics from first principles (see in the following section).

We note that bright solitons occur under the same condition as for MI to occur; namely for $PQ > 0$. Although this appears to be an analytical coincidence via this model, and the phenomena should be clearly distinguished (linear amplitude stability analysis in one case, nonlinear PDE theory in the other), it has been postulated in another context [39] that MI is the first evolutionary stage toward the formation of (a series of) envelope solitons. Envelope solitons form of the bright type are widely observed in abundance in space plasmas [9, 40, 45].

6. Numerical analysis

As discussed above, the modulation dynamics of the amplitude essentially depends on the coefficient ratio $r = P/Q$: positive values of $r$ correspond to MI occurrence and bright soliton existence, while negative $r$ leads to envelope stability and dark envelopes. The ratio $r$ determines the soliton width $L$ (for given amplitude $\psi_0$) as $L \sim r^{1/2}/\psi_0$, via (31), since higher $|r|$ values suggest wider (spatially extended) solitons and vice versa. Finally, the (perturbation wavenumber) threshold for instability to occur is $k_{ir} \sim r^{-1/2}$, as determined by (25): higher $r$ implies a lower threshold, i.e., a smaller instability ‘window’. A parametric investigation along these lines is carried out below, by investigating the behavior of $r$ against variations of
Figure 7. Variation of the NLSE coefficient ratio ($P/Q$) with carrier wave number $k$ for fixed $\beta = 0.5$ and for different values of $\kappa$ (as indicated on each curve in the plot).

Figure 8. The $P/Q$ coefficient ratio, whose sign is related to the type of solitary excitations, is depicted against the cold-to-hot electron density ratio $\beta$ for $k = 3$ (for different values of $k$, as indicated on each curve in the plot).

6.1. Parametric investigation

Dependence on the carrier wavenumber. In order to investigate the stability profile we have depicted the ratio $r = P/Q$ versus the carrier wavenumber $k$ in figure 7 for different values of $\kappa$. Note the pole(s) (corresponding to roots of $Q$), where the ratio changes sign determine the threshold $k_{cr}$, above which the instability sets in. In other words, the threshold $k_{cr}$ separates the stable region(s) ($k < k_{cr}$ or $P/Q < 0$) from the unstable one(s) ($k > k_{cr}$ or $P/Q > 0$). We see that the critical value of the wavenumber $k$ increases with excess superthermality, i.e., for lower $\kappa$. Dark-type solitons therefore exist for the small values of the carrier wavenumber $k$, while bright solitons exist for higher $k$ (or shorter wavelengths). Observing, e.g., the region $2.5 \leq k \leq 3.3$ in figure 7, we predict that bright solitons, say occurring for Maxwellian warm electrons, may be destabilized and will cease to exist for lower $\kappa$ values.

Dependence on the plasma concentration. In figure 8, we have depicted the ratio $r = P/Q$ versus the cold-to-hot electron number density ($\beta$) for different values of $\kappa$. We point out the existence of a $\beta$ threshold beyond which $r$ changes sign, i.e., bright solitons appear to exist...
Figure 9. Dependence of the critical carrier wavenumber (instability threshold) \(k_{cr}\) on (a) excess superthermality via \(\kappa\) we depict the contour \(Q = 0\) for different values of \(\beta\); (b) the cold-to-hot electron density ratio \(\beta\) (i.e., the critical wave number threshold is shown for different \(\kappa\)).

for low values of \(\beta\) while dark-gray ones occur for higher values of \(\beta\). Also, modulational stability is ensured for high values of \(\beta\) (cold electrons dominant), while MI occurs for lower \(\beta\) (hot electrons dominant). The critical value of \(\beta\) is lower for higher superthermality, which therefore somehow suppresses MI, i.e., stable regions for high \(\kappa\) may be destabilized for lower \(\kappa\) (see the area for \(\beta\) between 0.4 and 1.6, roughly, in figure 8).

The effects discussed in the latter two paragraphs are elegantly combined into figure 9, where the wave number threshold \(k_{cr}\) dependence on \(\kappa\) (left panel, for different values of \(\beta\)) and on \(\beta\) (right panel, for different values of \(\kappa\)) is made apparent. The curves shown in figure 9 represent the contours \(PQ = 0\); recall that the left regions (small \(k\)) correspond to stable regions, while the right ones (higher \(k\)) are modulationally unstable. Figure 9 suggests that (for a fixed value of \(\beta\)) the wave number threshold \(k_{cr}\) increases as \(\kappa\) decreases. The existence of bright- or dark-type soliton is thus also determined by the level of superthermality (via \(\kappa\)); note the different curves in figure 9(b). From figure 9 it is obvious that \(k_{cr}\) increases with increasing \(\beta\) (for fixed \(\kappa\)), i.e., the stability window grows wider by increasing the hot electron number density. As a consequence, bright-type solitons may be eliminated (destabilized) by increasing \(\beta\) (for fixed \(\kappa\)). Furthermore, a larger stability window is provided by decreasing superthermality. The effect of the superthermal electron component on solitary structures is therefore significant.

The square root of the \(P/Q\) ratio provides the width \(L\) of bright-type envelope solitons (for given maximum amplitude), as explained in detail in [41, 40]; cf equation (31). To trace the effect of superthermality on \(L\), we have depicted the value of \(\sqrt{\frac{P}{Q}}\) with \(\beta\) (cold-to-hot electron number density) in figure 10. It is clear that higher values of \(P/Q\), leading to wider solitons, are here obtained for higher \(\beta\) (for fixed \(\kappa\)) and for lower \(\kappa\) (for fixed \(\beta\)).

As expected, our results for large \(\kappa\) recover exactly those of [41], where a Maxwellian electron distribution was assumed. For instance, by finding the roots of \(Q\) numerically, we find the asymptotic value(s) \(k_{cr} = 2.47\) for \(\beta = 0.5\), and also \(k_{cr} = 2.71\) for \(\beta = 1\) (see figure 9), for infinite large values of \(\kappa\), which agrees exactly with figures 1 and 2, respectively, of [41] (setting \(\theta = 0\) therein, i.e., looking at the intersection with the horizontal axis).
Figure 10. Variation of the envelope soliton width $L$ (for a given amplitude $\psi_0$) with the cold-to-hot electron density ratio $\beta$. We have taken $k = 3$ here. Recall that the width of bright-type solitary excitations for given amplitude $\psi_0$ is given by equation (31).

6.2. Numerical stability of the soliton solutions of the NLSE

In order to test our predictions for the nature and stability of envelope solitons (as given by equations (29) and (30) above), we have integrated the NLSE numerically. We have adopted a hypothetical scenario, in which a bright soliton propagates in a Maxwellian plasma, and then encounters a superthermal region, where the same solution would be anticipated to be unstable. Therefore, we have chosen a representative set of values ($k = 3, \beta = 0.5$), which allow for both cases of interest, depending on superthermality. Indeed, as obvious from figure 7, a wavepacket (with carrier wave, e.g., here, $k = 3$) would be characterized by a positive value of $P/Q$ for large $\kappa$ (e.g., $\kappa = 100$), while it would correspond to a negative value of $P/Q$ for small $\kappa = 3$ (to see this, compare the solid and dashed curves in figure 7).

We have integrated the NLSE via a Runge Kutta 4 method, in the two cases described above: first, assuming quasi-Maxwellian electrons (for $\kappa = 100$) and then for strongly superthermal electrons (for $\kappa = 3$), while keeping all other parameters fixed ($k = 3, \beta = 0.5$). The exact soliton expression (30) was taken as initial condition in both cases. The result is depicted in figures 11 and 12, respectively. As expected, the envelope profile remains stationary in the Maxwellian case; see figure 11 (disregarding a slight ‘breathing’ effect, due to a nonlinear phase change of the exact solution). In the presence of a superthermal electron component, on the other hand, the envelope is seen to spread and eventually decay, stretching its energy in space: see figure 12. This effect is also nicely depicted in figure 13(b) (cf figure 13(a), in the stable case).

The inverse trend is manifested by a dark soliton. For $\kappa = 3$, we have $PQ < 0$ (see figure 7) and dark solitons are ‘normal’ nonlinear modes in our system. Indeed, a dark-type initial condition propagates in a stable manner: see figure 14. However, for or $\kappa = 100$, we have $PQ > 0$, and the system favors bright-type solutions. Interestingly, in figure 15, a dark-type envelope soliton launched as initial condition becomes unstable, and localizes its energy by decomposing into a series of localized envelope pulses. This behavior is depicted on figure 16(b) (cf 16(a), for $PQ < 0$, where the same initial condition is stable). This is in perfect agreement with our predictions based on figure 7.

We have completed our study by considering a harmonic initial condition in the form $\psi(0, \zeta) = \sqrt{2 |P/Q|} \exp(i\zeta/2P)$. Taking $\kappa = 100$ (quasi-Maxwellian background), the wavepacket is expected to be modulationally unstable. Indeed, as shown in figure 17 energy
localization occurs after approximately 200 time units, resulting in the formation of a train of localized peaks (pulses), presumably evolving toward a bright soliton profile.

7. Conclusions

We have undertaken a thorough investigation, from first principles, of the existence and stability of electron-acoustic envelope excitations and the associated modulational instability of electrostatic wavepackets. We have considered a three-component plasma consisting of a cold electron fluid, hot electrons obeying the kappa type distribution, and stationary ions. Relying on a nonlinear Schrödinger equation derived for the electric potential, we have computed exact
expressions for the dispersion ($P$) and nonlinearity ($Q$) coefficients, and have investigated their parametric dependence on the main two parameters of relevance, namely the cold-to-hot electron density ratio $\beta$ and the ‘spectral index’ $\kappa$ (which measures the excess superthermality of the electron background).

We have pointed out the dependence on $\kappa$ of all linear and nonlinear quantities of interest, including the phase velocity (lower for nonthermal plasmas than in Maxwellian plasmas) and the critical carrier wavenumber $k_{cr}$, above which modulational instability sets in (while smaller $k$, i.e., longer wavelengths, correspond to stable wavepackets). The threshold $k_{cr}$ increases with lower $\kappa$ (deviation from Maxwellian df for the electrons), while it attains a constant value for higher $\kappa$, as expected (Maxwellian limit). We also find that $k_{cr}$ decreases with decreasing values of $\beta$, suggesting that modulational instability sets in easier (i.e., for longer wavelengths) with an increase in the cold electron component.
The modulational instability growth rate was studied for different values of $\kappa$. The instability growth rate is reduced due to excess superthermality. The dominance of the cold electron population also affects the modulational instability growth rate by reducing the growth rate, i.e., somehow controlling the instability.

For a given $\beta$ value, an increase in superthermality leads to wider bright-type envelope solitons at constant amplitude. For given $\kappa$ on the other hand, the soliton width increases with an increasing $\beta$, i.e., considering more cold electrons leads to wider bright-type solitary excitations.

Our theoretical predictions were confirmed by a series of numerical simulations, which have established the stability of bright (or dark, separately) soliton modes in the positive (or negative, respectively), $PQ$ product case(s). On the opposite case(s), the same modes as initial conditions were found to be unstable. Electrostatic energy localization via modulational instability occurs in the plasma, as a harmonic wavepacket initially launched as initial condition...
was shown to evolve toward a localized multi-soliton state. Interestingly, the same trend was witnessed for dark-type envelopes.

Our theoretical investigation aims at contributing to understanding the basic features of localized electrostatic solitary waves (envelope excitations) in space and laboratories plasmas, where energetic (accelerated) particles occur.

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Appendix: Harmonic amplitudes obtained via multiscale perturbation theory

The amplitudes corresponding to the first harmonics in order $\epsilon^2$ are given by

$$n_i^{(2)} = \frac{2ik}{\beta} \frac{\partial \phi_i^{(1)}}{\partial X_1}, \quad u_i^{(2)} = \frac{i\omega}{\beta} \frac{\partial \phi_i^{(1)}}{\partial X_1}.$$  

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For \( n = 2 \) and \( l = 2 \), the evolution equations provide the amplitudes of the second order harmonics which are found to be proportional to \( (\phi_1^{(1)})^2 \):

\[
\begin{align*}
n_2^{(2)} &= c_1^{(22)} (\phi_1^{(1)})^2, \\
u_2^{(2)} &= c_2^{(22)} (\phi_1^{(1)})^2, \\
\phi_2^{(2)} &= c_3^{(22)} (\phi_1^{(1)})^2.
\end{align*}
\]  

(33)

where

\[
\begin{align*}
c_1^{(22)} &= -\frac{(4k^2 + c_1)}{\beta} c_3^{(22)} - \frac{c_2}{\beta}, \\
c_2^{(22)} &= \omega k \left[ \frac{(k^2 + c_1)^2}{2\beta v_g^2} + \frac{(k^2 + c_1)^2}{2\omega^2 \beta} + \frac{c_2}{3\omega^2} \right], \\
c_3^{(22)} &= -\frac{c_2}{3k^2} - \frac{(k^2 + c_1)^2}{2k^2 \beta}.
\end{align*}
\]

(34)

We also obtain (combining the expressions for \( n = 2, l = 0 \) and \( n = 3, l = 0 \))

\[
\begin{align*}
n_0^{(2)} &= c_1^{(20)} |\phi_1^{(1)}|^2, \\
u_0^{(2)} &= c_2^{(20)} |\phi_1^{(1)}|^2, \\
\phi_0^{(2)} &= c_3^{(20)} |\phi_1^{(1)}|^2,
\end{align*}
\]

(35)

where

\[
\begin{align*}
c_1^{(20)} &= \frac{1}{v_g} c_2^{(20)} + \frac{2c_1}{\beta v_g^2}, \\
c_2^{(20)} &= \frac{1}{v_g} \left( \frac{k^2 + c_1}{\beta} - c_3^{(20)} \right), \\
c_3^{(20)} &= \frac{k^2 + 3c_1 + 2v_g^2 c_2}{\beta - v_g^2 c_1}.
\end{align*}
\]

(36)

References

[8] Baluku T K and Hellberg M A 2010 private communication University of Kwa-Zulu Natal Durban S. Africa