Ion-acoustic waves in a plasma consisting of adiabatic warm ions, non-isothermal electrons and a weakly relativistic electron beam: linear and higher-order nonlinear effects

A. Esfandyari-Kalejahi\textsuperscript{1}, I. Kourakis\textsuperscript{2,3} \textsuperscript{*} and P. K. Shukla\textsuperscript{2}

\textsuperscript{1} Azarbaijan University of Tarbiat Moallem, Faculty of Science, Department of physics, 51745-406, Tabriz, Iran

\textsuperscript{2} Institut für Theoretische Physik IV, Fakultät für Physik und Astronomie, Ruhr–Universität Bochum, D-44780 Bochum, Germany

\textsuperscript{3} Center for Plasma Physics (CPP), Department of Physics and Astronomy, Queen’s University Belfast, BT7 1 NN Northern Ireland, UK

(Dated: Accepted 2 January 2008)

\textsuperscript{*} Work initiated while at: Universiteit Gent, Sterrenkundig Observatorium, Krijgsstraat 281, B-9000 Gent, Belgium
Abstract

The nonlinear propagation of finite amplitude ion acoustic solitary waves in a plasma consisting of adiabatic warm ions, non-isothermal electrons and a weakly relativistic electron beam is studied via a two-fluid model. A multiple scales technique is employed to investigate the nonlinear regime. The existence of the electron beam gives rise to four linear ion acoustic modes, which propagate at different phase speeds. The numerical analysis shows that the propagation speed of two of these modes may become complex-valued (i.e. solitary waves cannot occur) under conditions which depend on values of the beam-to-background-electron density ratio $\alpha$, the ion-to-free-electron temperature ratio $\sigma$ and the electron beam velocity $v_0$; the remaining two modes remain real in all cases. The basic set of fluid equations are reduced to a Schamel-type equation and a linear inhomogeneous equation for the first and second-order potential perturbations, respectively. Stationary solutions of the coupled equations are derived using a renormalization method. Higher-order nonlinearity is thus shown to modify the solitary wave amplitude and may also deform its shape, even possibly transforming a simple pulse into a W-type curve for one of the modes. The dependence of the excitation amplitude and of the higher-order nonlinearity potential correction on the parameters $\alpha$, $\sigma$ and $v_0$ is numerically investigated.

PACS numbers: 05.45.Yv, 52.35.Sb, 52.35.Fp

Keywords: Ion acoustic waves, non-isothermal plasma, electron beam, Schamel equation, reductive perturbation technique, renormalization method.
I. INTRODUCTION

The nonlinear propagation of electrostatic (ES) and electromagnetic (EM) excitations in plasmas has received considerable attention in the last few decades, as witnessed by the increasing number of relevant monographs (see for example Refs. [1–3]). Spacecraft missions have enabled in-situ measurements of electric and magnetic fields e.g. in the magnetosphere and beyond as shown in Refs. [3, 4]. Space observations provide abundant information about stationary nonlinear ES/EM structures occurring in space. Such localized ES structures may, for instance, be responsible for the acceleration of auroral particles to KeV energies see Ref. [5] and references therein. A number of localized ES excitations traced e.g. by the GEOTAIL mission have been reported to propagate at speeds much slower than the electron thermal speed, and have thus been effectively interpreted as ion-acoustic (IA) wave related nonlinear excitations (see for example Ref. [6]), while subsequent theoretical (see Ref. [7]) and ab-initio numerical, as given in Refs. [8, 9], investigations have confirmed those results.

Of particular interest is the case when streaming particles are injected in plasmas, where they often evolve towards a coherent trapped-particle state, as has been confirmed by experiments, see Ref. [10]. A theoretical description of this state in terms of electron holes has been provided in Ref. [11]. The onset of electron trapping is also observed in relation with the formation of double layers (see in Ref. [12] for a review) and in computer simulations, as reported in Ref. [13]. The presence of trapped particles can significantly modify the wave propagation characteristics in collisionless plasmas, as shown in Refs. [14, 15]. In particular, mode propagation in the presence of resonant particles for phase speeds near the thermal range is intrinsically nonlinear and is thus generally characterized by both wavenumber- and amplitude-dependent dispersion relations; see in Refs. [16, 17]. Other aspects of trapped-particle modes (namely involving phase space vortices) have also been studied by several authors; see, e.g., in Refs. [18–20].

The presence of an electron beam in plasmas is known to be associated with various interesting effects, including e.g., nonlinear wave amplification, investigated in Ref. [21]. Electron beams are typically encountered in the upper layers of the magnetosphere, as reported by satellite missions, e.g. the FAST mission at the auroral region (see Refs. [22–25]), the S3-3 (see Ref. [26]), Viking (see Ref. [27]), and the GEOTAIL and POLAR (see Refs. [25, 28–30]) missions. Such a beam-plasma system has also been created in laboratory
From a nonlinear plasma-theoretical point of view, the existence of an electron beam in a plasma has been shown to modify the properties and conditions for the existence of nonlinear excitations, including arbitrary amplitude localized ES excitations as given in Refs. [34–37] (typically modelled via the Sagdeev pseudopotential formalism) or small amplitude solitary pulses, as in Refs. [38–41] (i.e. generically related to Korteweg-de Vries (KdV) and related equation theories). Interestingly, a novel \((1 + 1/2)\)-nonlinearity was shown to be associated with electron trapping in Refs. [14, 15], modelled by a hybrid Schamel-KdV-type nonlinear evolution equation [see (49) in [14]], in contrast with the quadratic and cubic nonlinearity respectively present in the KdV and modified KdV (mKdV) equations. The investigation in hand may in fact be viewed as an extension of Schamel’s work in Refs. [14, 15]. Of interest are also predictions of modulated envelope wavepackets, typically modelled as solutions of a nonlinear Schrödinger (NLS) type equation; see Refs. [42, 43]. Such theoretical considerations have later been extended to (higher-frequency) electron-acoustic waves (related to Broadband Electrostatic Noise, BEN, in the Earth’s auroral region; see in [44]), supported by numerical simulations (see Ref. [45]), and have also been considered with respect to ion-beam effects as given in Ref. [46]. Interestingly, pair (electron-positron) beam-plasma experiments have shown that, by transmitting a low-energy electron beam through positron plasmas stored in a Penning trap, large amplitude oscillations of the positron plasma were excited and even positron ejection was observed (see Ref. [47]).

The generic nonlinear plasma evolution equations, e.g. the KdV or the NLS equation(s) cited above, are obtained via appropriate multiple scales (reductive) perturbation techniques. Naturally, the know-how of these techniques was subsequently advanced to higher perturbation orders. Higher-order nonlinear contributions to the KdV equation were considered in Refs. [39, 40] and, in a more formal mathematical context, in Refs. [48–54].

In this work, we shall consider higher order nonlinear effects on the propagation of ion-acoustic solitary waves in a collisionless plasma consisting of warm ions, non-isothermal electrons and a weakly relativistic electron beam. Propagation parallel to the external magnetic field is considered.

The layout of this manuscript is as follows. In section II, the basic set of fluid equations are presented and reduced to a Schamel-type nonlinear equation for the first-order poten-
tial, coupled to a linear inhomogeneous equation for the second-order potential correction. Stationary solutions of the coupled equations, retaining terms up to the third order in the perturbation, are obtained in Section III. Results, discussions, and also numerical analysis are given in Sec. IV. Finally, section V devotes to the conclusions.

II. BASIC EQUATION AND FORMULATION

The basic set of (reduced) fluid equations for ions and the electron beam read

$$\frac{\partial n}{\partial t} + \frac{\partial (nv)}{\partial x} = 0,$$  \hspace{1cm} (1)

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{3\sigma}{(1 + \alpha)^2} n \frac{\partial n}{\partial x} + \frac{\partial \phi}{\partial x} = 0,$$  \hspace{1cm} (2)

$$\frac{\partial n_b}{\partial t} + \frac{\partial (n_b v_b)}{\partial x} = 0,$$  \hspace{1cm} (3)

$$\left(\frac{\partial}{\partial t} + v_b \frac{\partial}{\partial x}\right) (\gamma_b v_b) - \frac{1}{\mu_i} \frac{\partial \phi}{\partial x} = 0.$$  \hspace{1cm} (4)

(following the fluid model in Ref. [36]), where $\gamma = \sqrt{1 - \left(\frac{v_b}{c}\right)^2}$ (c is the velocity of light in vacuum). Here $\sigma = T_i/T_{ef}$, $\alpha = n_{b0}/n_0$ and $\mu = m_e/m_i$, where $T_i$, $n_0$ ($n_{b0}$) and $m_e$ ($m_i$) are the ion temperature, the unperturbed background electron (electron beam) density and the electron (ion) mass respectively. The variables $n_e$, $n_b$, $n$, $v_b$, $v$ and $\phi$ denote the electron density, the electron beam density, the ion density, the beam electron velocity, the ion velocity and the electrostatic potential, respectively. The velocities are normalized by the ion sound speed $v_s = (k_B T_{ef}/m_i)^{\frac{1}{2}}$, the time $t$ and the distance $x$, by the ion plasma frequency $\omega_{pi}^{-1} = [\epsilon_0 m_i/(n_0 e^2)]^{\frac{1}{2}}$ and the electron Debye length $\lambda_d = [\epsilon_0 k_B T_{ef}/(n_0 e^2)]^{\frac{1}{2}}$, the densities by the background electron density $n_0$, and the potential by $k_B T_{ef}/e$, where $e$ is the charge of the electron.

The system is closed by Poisson’s equation

$$\frac{\partial^2 \phi}{\partial x^2} = n_e + n_b - n.$$  \hspace{1cm} (5)

Following the Schamel model for non-isothermal electrons [16, 17], we shall assume that the electron density is given by

$$n_e = 1 + \varphi - \frac{4}{3} b \varphi^2 + \frac{1}{2} \varphi^2 + \ldots,$$  \hspace{1cm} (6)
(see for example also in Refs. [14, 15]), in which \( b = (1 - \beta)/\pi \frac{1}{2} \) and \( \beta = T_{ef}/T_{et} \). Here and above, \( T_{ef} \) denotes the (constant) temperature of free electrons and \( T_{et} \) that of trapped electrons. The third term in relation (6) introduces the contribution of the resonant electrons to the electron density. The case \( \beta = 1 \) (or \( b = 0 \)) corresponds to the case of isothermal electrons which has already been studied via the pseudopotential method in Ref. [36] and also by reductive perturbation techniques (yet to the lowest order in nonlinearity) in Ref. [41]. According to the above scaling, \( 1 + \alpha, \alpha \) and \( v_0 \) are the unperturbed values of \( n, n_b \) and \( v_b \), respectively. Replacing \( n \) by \( 1 + \alpha + \tilde{n} \) and \( v_b \) by \( v_{b0} + \tilde{v}_b \), and adopting a weakly relativistic approximation, Eqs. (1)-(5) lead to a system of evolution equations for the dynamic state (displacement from equilibrium, tilde marked) variables. Henceforth dropping tildes everywhere, for simplicity, the dynamical evolution system becomes

\[
\frac{\partial n}{\partial t} + (1 + \alpha) \frac{\partial v}{\partial x} + \frac{\partial (nv)}{\partial x} = 0, \tag{7}
\]

\[
\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{3\sigma}{1 + \alpha} \frac{\partial n}{\partial x} + \frac{3\sigma}{(1 + \alpha)^2} n \frac{\partial n}{\partial x} + \frac{\partial \phi}{\partial x} = 0, \tag{8}
\]

\[
\frac{\partial n_b}{\partial t} + v_{b0} \frac{\partial n_b}{\partial x} + \frac{\partial (n_b v_b)}{\partial x} = 0, \tag{9}
\]

\[
\frac{\partial v_b}{\partial t} + v_{b0} \frac{\partial v_b}{\partial x} + v_b \frac{\partial v_b}{\partial x} - \frac{1}{\mu_i} \left(1 - \frac{3v_{b0}^2}{2c^2}\right) \frac{\partial \phi}{\partial x} + \frac{3v_{b0}v_b}{\mu_ic^2} \frac{\partial \phi}{\partial x} = 0, \tag{10}
\]

and

\[
\frac{\partial^2 \varphi}{\partial x^2} = n_e + n_b - n - 1. \tag{11}
\]

Note that, having dropped the tildes in Eqs. (7)-(11) (cf. the definitions preceding those equations), the equilibrium configuration corresponds to a zero value of all state variables in the latter equations.

In order to obtain an evolution equation describing the propagation of nonlinear ion-acoustic waves from the basic system of Eqs (6)-(11), we shall expand the densities, fluid velocities and electrical potential – by defining a smallness parameter \( \varepsilon \ (\ll 1) \) – near the equilibrium state:

\[
\begin{align*}
n & = \varepsilon n_1 + \varepsilon^2 n_2 + \varepsilon^3 n_3 + \varepsilon^4 n_4 + \ldots, \\
n_b & = \varepsilon n_{b1} + \varepsilon^2 n_{b2} + \varepsilon^3 n_{b3} + \varepsilon^4 n_{b4} + \ldots, \\
u & = \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \varepsilon^4 u_4 + \ldots, \\
v_b & = \varepsilon v_{b1} + \varepsilon^2 v_{b2} + \varepsilon^3 v_{b3} + \varepsilon^4 v_{b4} + \ldots, \\
\varphi & = \varepsilon \varphi_1 + \varepsilon^2 \varphi_2 + \varepsilon^3 \varphi_3 + \varepsilon^4 \varphi_4 + \ldots
\end{align*}
\]
and employ the stretched variables
\[ \xi = \epsilon^\frac{1}{4} (x - St), \quad \tau = \epsilon^\frac{3}{4} St \] (13)
(cf. Refs. [15]). For a discussion of the systematic method relying on the expansions (12) and the stretched variables (13) (first suggested in Ref. [14]), we refer to Refs. [2] and [3]. Subsequently, we substitute (12) and (13) into Eqs. (6)-(11) and isolate the coefficients of various powers in \( \epsilon \), equating the corresponding quantities.

To the lowest power in \( \epsilon \) we obtain
\[ v_1 = \frac{S}{S^2 - 3 \sigma} \varphi_1, \quad n_1 = \frac{1 + \alpha}{S^2 - 3 \sigma} \varphi_1, \quad v_{b1} = \frac{1 - \frac{3}{2} \frac{v_0^2}{c^2}}{(v_0 - S)^2} \varphi_1 \] (14)
along with the compatibility condition
\[ \frac{\alpha}{\mu} \left(1 - \frac{3}{2} \frac{v_0^2}{c^2}\right) + \frac{1 + \alpha}{S^2 - 3 \sigma} = 1, \] (15)
in complete agreement with the results in Ref. [41].

From the next-order equations in \( \epsilon \), we obtain the Schamel-type equation
\[ \varphi_{1\tau} + bA \varphi_1^{\frac{1}{2}} \varphi_{1\xi} + A \left(\frac{1}{2}\right) \varphi_{1\xi\xi\xi} = 0, \] (16)
where the subscripts denote partial differentiation, i.e. \((\cdot)_\xi = \partial(\cdot)/\partial \xi\) (and the same for \( \tau \)).

The dispersion coefficient \( A \) is
\[ A = \frac{(S^2 - 3 \sigma)^2(v_0 - S)^3}{S[(1 + \alpha)(v_0 - S)^3 - \frac{2}{\mu}(S^2 - 3 \sigma)^2(1 - \frac{3}{2} \frac{v_0^2}{c^2})]}. \] (17)
Note that the coefficient of the nonlinear term in (16) is equal to the non-thermality parameter \( b \) (and thus plainly cancels in the Maxwellian electron case, in which our results are therefore not valid). It may be pointed out that the \((1+1/2)\)-nonlinearity appearing in Eq. (16) was shown for the first time to be associated with electron trapping in Refs. [14, 15], where the scaling ansatz (13) was, in fact, introduced.

The second order quantities \( n_2, v_2, n_{b2}, v_{b2} \), and \( \varphi_2 \) can be expressed in terms of \( \varphi_1 \) and \( \varphi_2 \) as
\[ n_2 = -A \frac{S^2(1 + \alpha)}{(S^2 - 3 \sigma)^2} \varphi_{1\xi\xi} - \frac{4}{3} bA \frac{S^2(1 + \alpha)}{(S^2 - 3 \sigma)^2} \varphi_1^{\frac{3}{2}} + \frac{(1 + \alpha)}{(S^2 - 3 \sigma)} \varphi_2, \] (18)
\[ n_{b2} = -\frac{S\alpha A}{\mu} \left(1 - \frac{3 \frac{v_0^2}{c^2}}{2}\right) \varphi_{1\xi\xi} - 4S\alpha bA \left(1 - \frac{3 \frac{v_0^2}{c^2}}{2}\right) \varphi_1^{\frac{3}{2}} - \frac{\alpha}{\mu} \left(1 - \frac{3 \frac{v_0^2}{c^2}}{2}\right) \varphi_2, \] (19)
\begin{equation}
    v_2 = -\frac{2}{3} b A S \frac{(S^2 + 3\sigma)}{(S^2 - 3\sigma)^2} \varphi_1^2 - \frac{AS (S^2 + 3\sigma)}{2 (S^2 - 3\sigma)^2} \varphi_{1\xi\xi} + \frac{S}{(S^2 - 3\sigma)^2} \varphi_2, \tag{20}
\end{equation}

and

\begin{equation}
    v_{b2} = \frac{SA (1 - \frac{3\sigma^2}{S^2})}{2\mu (v_0 - S)^2} \varphi_{1\xi\xi} + \frac{2b S (1 - \frac{3\sigma^2}{S^2})}{3\mu (v_0 - S)^2} \varphi_1^2 + \frac{1}{\mu} \frac{(1 - \frac{3\sigma^2}{S^2})}{(v_0 - S)} \varphi_2. \tag{21}
\end{equation}

The subsequent order in \( \varepsilon \) yields the equation

\begin{equation}
    \varphi_2 + b A (\varphi_1^{\frac{1}{2}} \varphi_2)_\xi + \frac{A}{2} \varphi_{2\xi\xi} = \frac{A}{2} \left( P - bAQ \right) \varphi_1 \varphi_{1\xi} - \frac{A^2 Q}{4} \varphi_1^{\frac{1}{2}} \varphi_{1\xi\xi} - \frac{A^2 R b}{2} (\varphi_1^{\frac{1}{2}} \varphi_{1\xi\xi})_\xi - \frac{A^2 R}{4} \varphi_{1\xi\xi\xi\xi}, \tag{22}
\end{equation}

where

\begin{equation}
    P = 1 + \frac{3a(1 - \frac{3\sigma^2}{S^2})}{2\mu (v_0 - S)^2} + \frac{3b a(1 - \frac{3\sigma^2}{S^2})}{2\mu (v_0 - S)^2} - \frac{3(1 + \alpha)}{2(S^2 - 3\sigma)^2} (S^2 + 2\sigma), \tag{23}
\end{equation}

\begin{equation}
    Q = \frac{3b S A^2 (1 - \frac{3\sigma^2}{S^2})}{\mu (v_0 - S)^4} + \frac{3b S A^2 (1 + \alpha)(S^2 + \sigma)}{(S^2 - 3\sigma)^3}, \tag{24}
\end{equation}

and

\begin{equation}
    R = \frac{3a A S^2 (1 - \frac{3\sigma^2}{S^2})}{2\mu (v_0 - S)^4} + \frac{3A S^2 (1 + \alpha)(S^2 + \sigma)}{2(S^2 - 3\sigma)^3}. \tag{25}
\end{equation}

Summarizing our results so far, we see that the perturbative ansatz (12), in which the two leading order dynamical variables are related by expressions (14) and (18-21), respectively, along with the constraint (15) for the propagation speed \( S \), provides a solution of the plasma fluid model equations introduced above. The evolution of the system in time is thus described by Eqs. (16) and (22) (together with the accompanying coefficient definitions), the combination of which provides the electric potential up to order \( \varepsilon^{3/2} \) (neglecting \( \varepsilon^2 \) and higher orders). In the following, we shall see how the latter two evolution equations may be solved analytically.

### III. THE STATIONARY SOLUTION

Following the renormalization method of Kodama and Taniuti, as exposed in Refs. [48, 49] (also see Refs. [50–54]), Eqs. (16) and (22) can be modified as

\begin{equation}
    \frac{\partial \tilde{\varphi}_1}{\partial \tau} + b A \tilde{\varphi}_1^{\frac{1}{2}} \frac{\partial \tilde{\varphi}_1}{\partial \xi} + \frac{A}{2} \frac{\partial^3 \tilde{\varphi}_1}{\partial \xi^3} + \delta \lambda \frac{\partial \tilde{\varphi}_1}{\partial \xi} = 0, \tag{26}
\end{equation}

and

\begin{equation}
    \frac{\partial \tilde{\varphi}_2}{\partial \tau} + b A \frac{\partial (\tilde{\varphi}_1^{\frac{1}{2}} \tilde{\varphi}_2)}{\partial \xi} + \frac{A}{2} \frac{\partial^3 \tilde{\varphi}_2}{\partial \xi^3} + \delta \lambda \frac{\partial \tilde{\varphi}_2}{\partial \xi} = s(\tilde{\varphi}_1) + \delta \lambda \frac{\partial \tilde{\varphi}_1}{\partial \xi}, \tag{27}
\end{equation}

where

\begin{equation}
    s(\tilde{\varphi}_1) = \frac{1}{\mu} \frac{(1 - \frac{3\sigma^2}{S^2})}{(v_0 - S)} \tilde{\varphi}_1.
\end{equation}
where \( s(\tilde{\varphi}_1) \) here is the right hand side of Eq. (22). In Eqs. (26) and (27), the parameter \( \delta \lambda \) is introduced in such a way that the resonant term in \( s(\tilde{\varphi}_1) \) is canceled by the term \( \delta \lambda \frac{\partial \tilde{\varphi}_1}{\partial \xi} \) in (27). The reader is referred to the original Refs. [48, 49] (also see [52]), for details on the tedious mathematical formalism.

Following the methodology in Ref. [52], we shall seek stationary profile moving solutions of the latter two equations, by defining the new moving stationary frame variable \( \eta \) as

\[
\eta = \xi - (\lambda + \delta \lambda) \tau
\]

where the parameter \( \lambda \) is related to the Mach number \( M = V/v_s \) by \( \lambda + \delta \lambda = M - 1 = \delta M \). Here \( V \) is the propagation velocity of the anticipated localized excitation(s). Under this transformation, Eqs. (26) and (27) become

\[
A \frac{\partial^3 \tilde{\varphi}_1}{\partial \eta^3} + \frac{4bA}{3} \frac{\partial}{\partial \eta} \tilde{\varphi}_1^{\frac{3}{2}} - 2\lambda \frac{\partial \tilde{\varphi}_1}{\partial \eta} = 0,
\]

and

\[
A \frac{\partial^3 \varphi_1}{\partial \eta^3} + 2bA \frac{\partial}{\partial \eta} (\tilde{\varphi}_1^{\frac{3}{2}} \varphi_2) - 2\lambda \frac{\partial \varphi_2}{\partial \eta} = 2 \left[ s(\tilde{\varphi}_1) + \delta \lambda \frac{\partial \tilde{\varphi}_1}{\partial \eta} \right].
\]

Using the boundary conditions

\[
\tilde{\varphi}_1 = \tilde{\varphi}_2 = \frac{\partial \tilde{\varphi}_1}{\partial \eta} = \frac{\partial \tilde{\varphi}_2}{\partial \eta} = \frac{\partial^2 \tilde{\varphi}_1}{\partial \eta^2} = \frac{\partial^2 \tilde{\varphi}_2}{\partial \eta^2} = 0,
\]

as \( |\eta| \to \infty \)

we can integrate Eqs. (29) and (30) and obtain

\[
\frac{\partial^2 \tilde{\varphi}_1}{\partial \eta^2} + \left( \frac{4b}{3} \tilde{\varphi}_1^{\frac{1}{2}} - \frac{2\lambda}{A} \right) \tilde{\varphi}_1 = 0,
\]

and

\[
\frac{\partial^2 \varphi_1}{\partial \eta^2} + 2 \left( b \tilde{\varphi}_1^{\frac{1}{2}} - \frac{\lambda}{A} \right) \varphi_2 - \frac{2}{A} \int_{-\infty}^{\eta} \left[ s(\tilde{\varphi}_1) + \delta \lambda \frac{\partial \tilde{\varphi}_1}{\partial \eta} \right] d\eta.
\]

The solitary wave solution of Eq. (29) is given by

\[
\tilde{\varphi}_1 = \varphi_0 \sech^4 \left( \frac{\eta}{D} \right),
\]

in which

\[
\varphi_0 = \frac{225\lambda^2}{(8bA)^2},
\]

and

\[
D = \left( \frac{8A}{\lambda} \right)^{\frac{1}{2}}.
\]
Note that positivity of $A$ is assumed, for reality of the solutions to be ensured. It may be pointed out that relations (34)-(36) essentially reflect the pulse excitation form (as well as the exact amplitude-width dependence) originally obtained in Ref. [14] (also see [15]).

The right hand side of Eq. (33), therefore, takes the form

$$
\frac{2}{A} \int_{-\infty}^{\eta} \left[ s(\tilde{\phi}_1) + \delta \lambda \frac{\partial \tilde{\phi}_1}{\partial \eta} \right] d\eta = \\
\left[ \frac{1}{2} (P - bAQ) \varphi_0^2 + \frac{15}{2} \frac{AQ}{D^2} + \frac{42}{3} \frac{bAR}{D^2} \right] \varphi_0^3 - \frac{420}{D^5} AR \varphi_0 \sech^8 \frac{\eta}{D} \\
+ \left[ -\frac{16AQ}{3D^2} + \frac{12bAR}{D^2} \right] \varphi_0^3 + \frac{520AR}{D^4} \varphi_0 \sech^6 \frac{\eta}{D} + \\
\left( \frac{2}{A} \delta \lambda - \frac{16^2AR}{2D^4} \right) \varphi_0 \sech^4 \frac{\eta}{D} .
$$

(37)

Now, in order to cancel the secular terms in $s(\tilde{\phi}_1)$, we may set the third line in the right-hand side of the latter equation equal to zero. We thus immediately get

$$
\delta \lambda = \frac{64A^2R}{D^4} = R\lambda^2 .
$$

(38)

Taking Eq. (37) into account, Eq. (33) becomes

$$
\frac{\partial^2 \tilde{\phi}_2}{\partial \eta^2} + 2(b \tilde{\phi}_1^2 - \frac{1}{A}) \tilde{\phi}_2 = \\
\left\{ \frac{1}{2} (P - bAQ) \varphi_0^2 + \frac{15}{2} \frac{AQ}{D^2} + \frac{42}{3} \frac{bAR}{D^2} \right\} \varphi_0^3 - \frac{420}{D^5} AR \varphi_0 \sech^8 \frac{\eta}{D} \\
+ \left[ -\frac{16AQ}{3D^2} + \frac{12bAR}{D^2} \right] \varphi_0^3 + \frac{520AR}{D^4} \varphi_0 \sech^6 \frac{\eta}{D} \\
\left( \frac{2}{A} \delta \lambda - \frac{16^2AR}{2D^4} \right) \varphi_0 \sech^4 \frac{\eta}{D} .
$$

(39)

In order to solve Eq. (39), we introduce a new variable

$$
\mu = \tanh\left( \frac{\eta}{D} \right) .
$$

(40)

Under this transformation, Eq. (39) becomes

$$
\frac{d}{d\mu} \left[ (1 - \mu^2)^2 \frac{d\tilde{\phi}_2}{d\mu} \right] + [5(5 + 1) - \frac{4^2}{1 - \mu^2}] \tilde{\phi}_2 = \tau(\mu) ,
$$

(41)

where

$$
\tau(\mu) = (1 - \mu^2)^3 Z_1 + (1 - \mu^2)^2 Z_2 ,
$$

(42)

in which

$$
Z_1 = \frac{D^2}{2} (P - bAQ) \phi_0^2 + \frac{A}{2} (15Q + 56bR) \phi_0^3 - \frac{420}{D^2} AR \phi_0 ,
$$

(43)

and

$$
Z_2 = -A \frac{16}{3} Q + 24bR) \phi_0^3 + \frac{520}{D^2} AR \phi_0 .
$$

(44)
Equation (41) has two independent solutions in terms of associated Legendre functions, which are given by
\[ P_5^4(\mu) = 945\mu(1 - \mu^2), \]  
(45)
and
\[ Q_5^4(\mu) = \frac{945}{2}\mu(1 - \mu^2)\ln \frac{1 + \mu}{1 - \mu} - 334(1 - \mu^2)^2 
+ 975\mu^2(1 - \mu^2) + 630\mu^4 + 264\frac{\mu^6}{1 - \mu^2} + 48\frac{\mu^8}{(1 - \mu^2)^2} \].  
(46)
The particular solution of Eq. (41) can be written, by using the method of variation of parameters, as
\[ \tilde{\phi}_2(\mu) = u_1(\mu)P_5^4(\mu) + u_2(\mu)Q_5^4(\mu), \]  
(47)
where
\[ u_1 = -\frac{1}{945\times384}\left\{ -\left(\frac{407}{4}Z_1 + \frac{193}{2}Z_2\right)\mu + \left(\frac{3335}{12}Z_1 + \frac{3555}{15}Z_2\right)\mu^3 
- \left(\frac{843}{2}Z_1 + \frac{1344}{5}Z_2\right)\mu^5 + \left(\frac{693}{2}Z_1 + \frac{382}{6}Z_2\right)\mu^7 
- \left(\frac{595}{4}Z_1 + \frac{375}{12}Z_2\right)\mu^9 + \frac{105}{4}Z_1\mu^{11} 
- \left[\frac{105Z_1}{8}(1 - \mu^2)^6 + \frac{63Z_2}{4}(1 - \mu^2)^5\right] \ln \frac{1 + \mu}{1 - \mu}\right\}, \]  
(48)
and
\[ u_2 = -\frac{1}{384}\left[\frac{Z_1(1 - \mu^2)^6}{12} + \frac{Z_2(1 - \mu^2)^5}{10}\right]. \]  
(49)
The complementary solution of Eq. (41) is given by
\[ \tilde{\phi}_{2c} = c_1P_5^4(\mu) + c_2Q_5^4(\mu). \]  
(50)
Here, the first term is the secular one which can be eliminated by renormalizing the amplitude. Also, \( c_2 = 0 \) as a result of vanishing boundary conditions for \( \tilde{\phi}_2(\eta) \) as \( |\eta| \to \infty \). Therefore, only the particular solution (47), contributes to \( \tilde{\phi}_2(\eta) \) so that the stationary solution for the potential for ion-acoustic wave is given by
\[ \tilde{\phi}(\eta) = \tilde{\phi}_1(\eta) + \tilde{\phi}_2(\eta) 
= \phi_0\text{sech}^4\left(\frac{\eta}{B}\right) + \left[\left(\frac{1}{6}Z_1 + \frac{1}{10}Z_2\right)\text{sech}^4\left(\frac{\eta}{B}\right) - \frac{Z_1}{12}\text{sech}^6\left(\frac{\eta}{B}\right)\right]. \]  
(51)
Notice that we have absorbed the (small) multiplicative factors \( \epsilon^1 \) and \( \epsilon^{3/2} \) preceding \( \phi_1 \) and \( \phi_2 \) in (12e), and have also neglected higher orders.
IV. NUMERICAL ANALYSIS AND DISCUSSION

The dispersion relation (15) is quartic in $S$, and shows that the combined effect of a finite ion temperature, non-isothermal electrons and a relativistic electron beam, gives rise to four nonlinear ion-acoustic modes propagating at different velocities. In the following, some numerical results for the above four ion-acoustic modes are presented and discussed.

As regards the beam velocity, we have considered $0.3 \leq v_0 \leq 3$ i.e. a weakly relativistic case where the ion acoustic speed $v_s$ has been chosen equal to $10^7$ m/s and consequently $c = 30$ (in the scaling defined above). On the other hand, the equilibrium electron beam density $\alpha$ will be assumed to be very small with respect to that of background ions, so that the electric current due to the electron beam may be ignored. Furthermore, we shall take $b = -0.4$, i.e. the free electron temperature is higher than that of trapped electrons; see Eq. (6) and the discussion following it.

Figure 1 shows the roots of the relation (15) against $v_0$, for $\alpha = 10^{-5}$ (a similar plot is presented in Fig. 6, for $\alpha = 10^{-4}$). We remark that two real modes in $S$ exist for all values of $v_0$, while the remaining two modes may become imaginary for some values of $v_0$ (thus, no ion acoustic solitary waves propagate in this case). Note that one of the real modes possesses negative values, suggesting soliton propagation in the direction opposite to the direction of the beam.

Now let us consider the effect of higher order nonlinearity on the above four modes. We note that according to the underlying assumptions of the perturbation theory adopted here (see Ref. [55]), the following condition must be satisfied

$$ \frac{|\phi_2|}{|\phi_1|} \leq 1 \quad (52) $$

(this requirement expresses, physically, what is expressed by the ordering in $\varepsilon$ above), so we have to choose appropriate values for the solitary excitation velocity $\lambda$; see Eq. (28). Therefore, we shall avoid too high values of the quantity $P$; see Eq. (23). Consequently, $Z_1$ and $\phi_2$ are given by Eq. (43) and Eq. (47), respectively, as provided by the numerical analysis.

In the case of the first root of Eq. (15) (see branch 1 in Fig. 1), Figure 2a shows $\frac{|\phi_2|}{|\phi_1|}$ at $\eta = 0$ versus $\lambda$, for different values of the beam velocity $v_0$, and for fixed values of $\alpha$ and $\sigma$. It is seen that the inequality (52) is satisfied for smaller values of $\lambda$, as $v_0$ increases. Figures
2(b) and (c) provide the form of the lowest-order solitary excitation - given by Eq. (34) - alone, and then in combination with the second-order one solution - from Eq. (51) - for arbitrary values of $\lambda = 0.01$, $\alpha = 10^{-5}$ and $\sigma = 0.5$, for two different values of $v_0$. It is clear that the effect of the second order nonlinearity leads to a decrease in the amplitude of the ion-acoustic solitary wave and this effect becomes more pronounced as the beam velocity increases. The second-order solution may have a W-type shape, with the central maximum having a positive value of $\phi$, for high values of the beam velocity (see Fig. 2c). It must be noted that the negative potential values appearing in Fig. 2c are due to the significant negative contribution of $\phi_2$ to the total $\phi$. According to condition (52), however, these values may be forbidden (read the discussion given by El-Taibany and Moslem in Ref. [56]).

Referring to mode 2 in Fig. 1, Figures 3(a-c) provide the same information as in Fig. 1, discussed above. Contrary to the first branch, the amplitude of the ion-acoustic excitation increases due to higher-order nonlinearity, and this effect becomes more intense as the beam velocity $v_0$ (solitary pulse velocity $\lambda$) increases (decreases).

For the third mode (branch 3, in Fig.1), there is no solitary wave for certain values of $\alpha$, since Eq. (15) then bears no real solution. Furthermore, see that the coefficient $A$ - Eq. (17) - then becomes negative, so that the excitation width - Eq. (36) - would then be imaginary since $\lambda$ is meant to be positive; no solitary excitation then exists. Nevertheless, one may consider negative values of $\lambda$ as well. This means that the lowest-order pulse excitation, as given by Eq. (34), will propagate in the negative space direction, opposite to that of the previous two modes. Figures 4(a)-(c) investigate the characteristics and solitary-wave solutions of the third mode. It is remarked that the amplitude of the ion-acoustic pulse increases due to higher-order nonlinear effects, in particular as the beam velocity $v_0$ (solitary wave velocity $\lambda$) increases (decreases). As Figs. 4b and 4c suggest, the normalized amplitudes are of the order of $10^{-13}$, in units of $K_B T_{eff} / e$, thus for $K_B T_{eff} \approx 10^7$, $\phi$ will be several micro-volts, which is meaningful practically. This is to justify a posteriori our choice to consider so small values of $\lambda$ (such as, for instance, $\lambda = 2 \times 10^{-8}$ in Fig. 5a).

The fourth mode (branch 4 in Fig.1) behaves in a similar manner to the third one, except for the fact that $\lambda > 0$ (see Figure 5), suggesting propagation along the positive axis direction.

In order to better investigate the role of the beam density, we have chosen a higher value of the (reduced) beam density $\alpha = 10^{-4}$, and have thus iterated our numerical considerations
for this value. A number of plots have thus been reproduced anew in this case, for $\alpha = 10^{-4}$.

The relation (15) is depicted in Fig. 6 (cf. Fig. 1). The contours of Eq. (52), the lowest-order pulse excitation - Eq. (34) - and the combined (to second-order) solution - Eq. (51) - for modes 1 to 4 have thus been depicted in Figs. 7 to 10 respectively (to be compared to Figs. 2-5, respectively, obtained for smaller $\alpha$). We observe that, although modes 2 to 4 provide qualitatively similar results for inequality (52), its analytical behavior is altered for mode 1. The effect of higher-order nonlinearity here leads to an increase in the amplitude of the localized excitation obtained within mode 1, even more so as the beam velocity decreases (see Figs. 7b and 7c), in contrast with the smaller beam density ($\alpha = 10^{-5}$ case); cf. Figs. 2b and 2c).

Finally, we have considered the effect of ion temperature $\sigma$ on the profile modification due to higher-order nonlinearity, for fixed values of $v_0$, $\alpha$. Numerical analysis shows that the higher nonlinear effect may either lead to an increase or a decrease of the excitation amplitude, as $\sigma$ increases, for fixed values of $v_0$ and $\alpha$. Here, we have considered two examples. Choosing $\alpha = 10^{-5}$ and $v_0 = 0.5$, we have considered modes 1 and 2 only (see Figs. 11 - 14). Here, the amplitude of the lowest-order pulse - Eq. (34) - increases as $\sigma$ increases, and so does the total excitation, in the presence of higher-order nonlinearity. Furthermore, the effect of higher-order nonlinearity is reduced as $\sigma$ increases, for both 1 and 2 modes.

V. CONCLUSION

The nonlinear propagation of ion-acoustic solitary waves in a collision-less plasma consisting of adiabatic warm ions, non-isothermal electrons and weak relativistic electron beam was studied, by using the reductive perturbation method. The basic set of model equations was reduced to a Schamel-equation for the first order potential, in addition to a linear inhomogeneous equation for the second-order potential. Exact stationary solutions of the coupled equations were obtained using a renormalization method. This result is to be compared to the case of isothermal electrons, where one obtains an ordinary KdV equation. The existence of an electron beam gives rise to four different types of nonlinear ion acoustic excitations, which propagate at different phase speeds. The numerical analysis shows that the phase speed $S$ of two of these modes can be complex-valued (namely, solitary waves can-
not occur for these modes) and this depends strongly on the beam-to-background-electron density ratio ($\alpha$) and electron beam velocity ($v_0$) and partially on the ion-to-free-electron temperature ratio ($\sigma$). On the other hand, the two remaining modes provide real values of $S$ (and thus exist) for all values of $\alpha$, $\sigma$ and $v_0$. On the other hand, it was shown that the effect of higher-order nonlinearity may modify the solitary wave’s amplitude and even deform the electric potential excitation’s shape from a simple positive pulse to a W-type curve. This depends on which mode is considered, and also on the physical beam-plasma parameters $\alpha$, $\sigma$ and $v_0$. We note that an extension of this investigation should be possible, covering cnoidal (or snoidal) wave solutions (see, for examples Refs. [14, 16, 17, 19, 57–62] ), which should allow one to obtain a nonlinear dispersion relation in which the wave frequency depends on both the wave number and the wave amplitude. In this way one could study the intrinsic relationship to the Buneman and the beam instability by relying on the system of equations (6)-(11). This is a challenging direction, which nevertheless goes beyond the scope of the present study, and may be investigated in the future.

The present investigation may be significant in understanding the properties of ion-acoustic localized structures in space observations, as well as in plasma laboratory experiments. 

Acknowledgments

I.K. acknowledges support from the Deutsche Forschungsgemeinschaft (DFG) under the Emmy-Noether program (grant SH 93/3-1), during the advanced stages of this work. He also gratefully acknowledges funding from the FWO (Fonds Wetenschappelijk Onderzoek-Vlaanderen, Flemish Research Fund) as well as the hospitality of Sterrenkundig Observatorium, University of Gent (Belgium), where the initial stage of this work was carried out. Prof. Frank Verheest is warmly thanked for organizing that visit. The anonymous reviewer is acknowledged for his/her critical comments.


[45] L. Quanming and S. Wang, Proceedings of the ISSS-7, *The 7th International School/Symposium for Space Simulations*, University of Kyoto, Kyoto (March 26 - 31, 2005);


FIG. 1: The soliton propagation velocity $S$ is depicted versus the equilibrium beam velocity $v_0$. The branches 1, 2, 3 and 4 show the four roots of the polynomial relation (15) respectively. Here, $\alpha = 10^{-5}$ and $\sigma = 0.5$. 
FIG. 2: (a) The ratio $|\phi_2|/|\phi_1|$ for $\eta = 0$ is plotted versus the velocity $\lambda$; (b) the mKdV Eq. (34) soliton solution $\phi_1$ and the combined second-order soliton $\phi$ [as given by (51)] are depicted, in the case of mode 1 (branch 1 in Fig. 1). Here, $\lambda = 0.01$, $\alpha = 10^{-5}$ and $\sigma = 0.5$ and $v_0 = 0.5$; (c) Same as in (b), for $v_0 = 1$. 
FIG. 3: Same as in Fig. 2, for mode 2 (branch 2 in Fig. 1).
FIG. 4: Same as in Fig. 2, for mode 3 (branch 3 in Fig. 1).
FIG. 5: Same as in Fig. 2, for mode 4 (branch 4 in Fig. 1).
FIG. 6: The soliton propagation velocity $S$ is depicted versus the equilibrium beam velocity $v_0$. The branches 1, 2, 3 and 4 show the four roots of the polynomial relation (15) respectively. Here, $\alpha = 10^{-4}$ and $\sigma = 0.5$. 

\[\frac{72x606}{72x606}FIG. 6: The soliton propagation velocity S is depicted versus the equilibrium beam velocity v_0.\]

\[\frac{72x606}{72x606}The branches 1, 2, 3 and 4 show the four roots of the polynomial relation (15) respectively. Here, \alpha = 10^{-4} and \sigma = 0.5.\]
FIG. 7: (a) The ratio $|\phi_2/|\phi_1|$ for $\eta = 0$ is plotted versus the velocity $\lambda$; (b) the mKdV Eq. (34) soliton solution $\phi_1$ and the combined second-order soliton $\phi$ [as given by (51)] are depicted, in the case of mode 1 (branch 1 in Fig. 1). Here, $\lambda = 0.001$, $\alpha = 10^{-4}$, $\sigma = 0.5$ and $v_0 = 0.5$; (c) Same as in (b), for $v_0 = 1$. 

23
FIG. 8: Same as in Fig. 7, for mode 2 (branch 2 in Fig. 1).
FIG. 9: Same as in Fig. 7, for mode 3 (branch 3 in Fig. 1).
FIG. 10: Same as in Fig. 7, for mode 4 (branch 4 in Fig. 1).
FIG. 11: Mode 1: the ion-acoustic soliton is depicted, as represented by the mKdV $\phi_1$ - from Eq. (34) - in comparison to the total (to second-order) excitation $\phi$ - given by (51) - for $v_0 = 0.5$, $\alpha = 10^{-5}$, $\lambda = 5 \times 10^{-3}$, and for various values of the temperature $\sigma$. 
FIG. 12: Same as in Fig. 11, here for \( \lambda = 2 \times 10^{-4} \), in the case of mode 2.
FIG. 13: Same as in Fig. 11, here for $\lambda = 5 \times 10^{-3}$ and $v_0 = 1$, in the case of mode 3.
FIG. 14: Same as in Fig. 11, here for $\lambda \approx 0.1 \times 10^{-6}$ and $v_0 = 1$, in the case of mode 4.