Kinetic Theory for a Test-Particle in Magnetized Plasma

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Abstract

The derivation of a kinetic equation for a test-particle weakly interacting with an electrostatic plasma in thermal equilibrium, subject to a uniform external magnetic field, is considered. From the generalized master equation a Fokker–Planck-type equation (FPE) follows as a “markovian” approximation. Such an equation does not preserve the positivity of the distribution function. Applying an averaging technique developed in the theory of open systems, a correct FPE is derived.

1. Introduction

A number of studies in Non-Equilibrium Statistical Mechanics have been devoted to the study of the relaxation of a small subsystem close to (but not at) equilibrium weakly interacting with a heat bath. In general, these approaches rely on a (“non-markovian”) generalized master equation (GME) obtained to 2nd order in the interaction. A Fokker–Planck-type equation is then derived from the GME as a “markovian” approximation [1]. In general, such an equation comes out not to preserve the positivity of the distribution function \( f(x, v; t) \). This problem, which is rather generic (that is, present for any particular dynamical problem considered) has been pointed out in the theory of open quantum-mechanical systems where possible remedy was suggested in [2,3]. An analytical procedure introduced therein was recently tested in the magnetized plasma case and a modified plasma kinetic equation was obtained [4]. In this paper, the problem is briefly exposed in a general manner and then focused upon in the case of an electrostatic plasma in a uniform magnetic field.

2. Statistical description – a master equation

We consider a test-particle \( \Sigma \) surrounded by (and weakly coupled to) a homogeneous background system of \( N \) particles (the reservoir ‘R’), subject to an external force field. The equations of motion for the test-particle are:

\[
\dot{x} = v; \quad \dot{v} = \frac{1}{m} \left( F_\theta(x, v) + \lambda F_{\text{int}}(x, v; X_R) \right)
\]  

(1)

\( \{ (x, v) \equiv \{ x_1(t), x_2(t); X_R \equiv \{ X_j \} \)  

\[ = \{ x_j(t), v_j(t)\}, j = 1, 2, \ldots, N \).

\( F_0 \) is due to the external field (e.g. Lorentz force \( F_L = (e/c)(v \times B) \) in the magnetized plasma case). The interaction force \( F_{\text{int}} = -(\partial/\partial x) \sum V(\{ x - x_j \}) \), actually the sum of interactions between \( \Sigma \) and \( R \)– particles surrounding it, may be viewed as a random process (in fact a Gaussian process with vanishing mean-value), as the reservoir will be assumed to be in equilibrium. In the particular case of a uniform external magnetic field, the zeroth-order \( (-\lambda^2) \) problem of charged particle motion yields the well-known (heliocoidal) solution.

Let \( \rho = \rho(T_R \cup T_\Sigma) \) be the total phase-space distribution function. The test-particle reduced distribution function (rdf) \( f(x, v; t) = \int f(\xi, X_R; t) \rho(\{ X, X_R \}; t) \) obeys a non-markovian Master Equation:

\[
\begin{align*}
\frac{\partial f(x, v; t)}{\partial t} &= L_0 f(x, v; t) + \lambda^2 n \int_0^t d\tau \int_{\Gamma_1} dx_1 dv_1 L_1 U^{(0)}(\tau) L_I \phi_{\text{eq}}(v_I) f(x, v; t - \tau)
\end{align*}
\]

(2)

to 2nd order in the (weak) interaction; \( L_0 \) (\( L_I \)) is the single particle ‘free’- (binary interaction) Liouville operator:

\[
L_0 = \frac{\partial}{\partial x} - \frac{1}{m} F_\theta \frac{\partial}{\partial v}, \quad L_I = F_{\text{int}}(\{ x - x_1 \}) \left( \frac{1}{m} \frac{\partial}{\partial v} - \frac{1}{m} \frac{\partial}{\partial v_1} \right)
\]

(3)

\( n = n_s = (N_s/V) \) is the particle density (a summation over particle species \( s \) is to be understood where appropriate). \( U^{(0)}(t) = \exp(L_0 t) \) denotes the ‘free’ (collisionless) Liouville evolution operator. Note that the mean-field \( (V\text{lasov}) \) term, in order \( \lambda^2 \), disappears once we assumed the reservoir state to be in a homogeneous equilibrium state \( \phi_{\text{eq}}(v_I) \) (essentially a Maxwellian state \( \phi_{\text{Max}}(v_I) \)).

3. A quasi-markovian approximation

A standard ‘markovianization’ procedure consists in assuming that \( f(t) \approx e^{-\lambda^2 t^2} f(t) \equiv U^{(0)}(-\lambda t) f(t) \) thus taking into account the explicit solution (provided that one exists) of the 0-th order problem of motion (in the field), and then evaluating the kernel asymptotically:

\[
\begin{align*}
\frac{\partial f}{\partial t} - L_0 f &= n \int_0^\infty d\tau \int_{\Gamma_1} dx_1 dv_1 L_1 U^{(0)}(\tau) L_I U^{(0)}(-\lambda) \phi_{\text{eq}} f \\
&= C[f]
\end{align*}
\]

(4)

\( (f = f(x, v; t)) \). One thus obtains the 2nd order PDE:

\[
\begin{align*}
\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} + \frac{1}{m} F_0 \frac{\partial f}{\partial v} = \frac{\partial}{\partial v} \left[ A \frac{\partial}{\partial v} + G \frac{\partial}{\partial x} + \mu a \right] f
\end{align*}
\]

(5)

where all coefficients are explicit functions of \( \{ x, v \} \) and the external field considered (since the details of single particle
motion in the field have been taken into account). Note that
the rhs can be re-arranged into the form of a ‘diffusion’
equation:

\[
\begin{align*}
\text{rhs} &= \frac{\partial^2}{\partial q^2}(D_{\text{eff}} f) - \frac{\partial}{\partial q} \left( F_r f \right),
\end{align*}
\]

\[ (r, s = 1, \ldots, 6; q \equiv \{x, v\} \in \mathbb{R}^6; D = \begin{pmatrix}
0 & \frac{1}{2} G^T A \\
\frac{1}{2} G & A
\end{pmatrix}. \]

It is worth noting that relations (1) through (5) are generic,
that is valid for any specific test-particle problem formulated
as above; the particular aspects of the dynamical problem
considered will be reflected in the form of the coefficients
in (5). The point we want to stress is that equation (5) does
not preserve the positivity of the d.f., as the \( (2d \times 2d) \) 2nd order ‘diffusion’ matrix \( D \) is
not positive definite. This fact is not necessarily true in
the homogeneous case (i.e. \( f = f(v) \)). In particular, in
the case of a charged particle inside a uniform magnetic field
along \( \hat{z} \) one obtains: \( A_{11} = A_{22} = D_\downarrow(v) \geq 0, \quad A_{12} = -
A_{21} = D_\uparrow(v), A_{33} = D_\uparrow(v) \geq 0, \ A_3 = A_3 = 0 \ (i = 1, 2) \) and
\( G = G(v) \) of the same (cylindrical-symmetric) form (exact
expressions for the coefficients can be found in [4]); thus,
integrating over space \( \{x\} \) the second terms in both sides
of (5) cancel:

\[
\begin{align*}
\frac{d f}{d t} + \frac{e}{2m} (v \times B) \frac{d f}{d v} &= \left( \frac{\partial}{\partial v_z} + \frac{\partial}{\partial v^2_x} \right) [D_{\downarrow}(v) f] + \frac{\partial}{\partial v_z} [D_{\uparrow}(v) f] \\
&= -\frac{\partial}{\partial v_x} [F_x(v) f] - \frac{\partial}{\partial v_y} [F_y(v) f] - \frac{\partial}{\partial v_z} [F_z(v) f].
\end{align*}
\]

(6)
The 3 × 3 diffusion matrix \( A \) is readily proved to be positive
definite.

4. Towards a markovian approximation ...

In search for a correct markovian approximation, we have
considered the averaging operator:

\[ A_{\epsilon} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} d\tau \ U(-\tau) \cdot U(\tau) \]

which was applied to the rhs of Eq. (5). The (markovian)
evolution operator thus defined was first introduced in
the theory of quantum open systems [2,5] and later
implemented in classical systems with discrete Liouville
eigenvalues [3].

The result in the homogeneous plasma case coincides
with (6) above. In the general case, however, the change
is rather dramatic; for a single-species plasma we find

the equation:

\[
\begin{align*}
\frac{d f}{d t} + \frac{e}{mc} (v \times B) \frac{d f}{d v} &= \left( \frac{\partial^2}{\partial v_x^2} + \frac{\partial^2}{\partial v_y^2} \right) [D_{\downarrow}(v) f] \\
&+ \frac{\partial^2}{\partial v_z^2} [D_{\uparrow}(v) f] + \left( \frac{\partial^2}{\partial v_x \partial y} - \frac{\partial^2}{\partial v_y \partial x} \right) [2 \omega_1^2 D_{\downarrow}(v) f] \\
&+ \omega_1^2 \left[ Q(v) + D_\downarrow(v) \right] \left( \frac{\partial^2}{\partial v_x^2} + \frac{\partial^2}{\partial v_y^2} \right) f(x, v; t) \\
&- \frac{\partial}{\partial v_x} [F_x(v) f] - \frac{\partial}{\partial v_y} [F_y(v) f] - \frac{\partial}{\partial v_z} [F_z(v) f] \\
&+ \Omega^2 F_x(v) \frac{d f}{d x} - \Omega^2 F_y(v) \frac{d f}{d y} - \Omega^2 F_z(v) \frac{d f}{d z}
\end{align*}
\]

(8)
(note the difference in structure from (5)) which we have assumed
that \( f \neq f(z) \). All coefficients, actually functions of
\( \{ v_x, v_y \}; \Omega \) \( \{ a_1 \equiv (a_1^2 + a_2^2)^{1/2}, a_1 \equiv a_2 \} \) \( \forall a \in \mathbb{R}^3 \); \( \omega \equiv \frac{e a}{m c} \), can be analytically evaluated in a convenient reference
frame. Preservation of the positivity of \( f(x, v; t) \)
by Eq. (8) can be readily verified.

5. Conclusion

As a matter of fact, equation (6) has appeared in previous
works on homogeneous plasma [6]; the present work is
in full agreement with those results. Furthermore, equations
in the form of (5) have appeared in [7], yet the cross-
position-velocity terms have been neglected. In conclusion,
Eq. (8) provides a correct kinetic description, from first
principles, of the dynamics of magnetized plasma, (to second
order in the interaction). In the homogeneous case, the
well-known previous result is obtained; furthermore, in
the absence of external field, the Landau equation is
recovered. Realistic generalizations, taking into account
field-inhomogeneities and/or geometry, are certainly
imposed and related work is in progress.

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