Nonlinear theory of dust lattice mode coupling in dust crystals *

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Quasi-crystals formed by charged mesoscopic dust grains (dust lattices), observed since hardly a decade ago, are an exciting paradigm of a nonlinear chain. In laboratory discharge experiments, these quasi-lattices are formed spontaneously in the sheath region near a negative electrode, usually at a levitated horizontal equilibrium configuration where gravity is balanced by an electric field. It is long known (and experimentally confirmed) that dust-lattices support linear oscillations, in the longitudinal (acoustic mode) as well as in the transverse, in plane (acoustic-) or off-plane (optic-like mode) directions. Either due to the (typically Yukawa type) electrostatic inter-grain interaction forces or to the (intrinsically nonlinear) sheath environment, nonlinearity is expected to play an important role in the dynamics of these lattices. Furthermore, the coupling between the different modes may induce coupled nonlinear modes. Despite this evidence, the elucidation of the nonlinear mechanisms governing dust crystals is in a rather preliminary stage. In this study, we derive a set of (coupled) discrete equations of motion for longitudinal and transverse (out-of-plane) motion in a one dimensional model chain of charged dust grains. In a continuum approximation, i.e. assuming a variation scale which is larger than the lattice constant, one obtains a set of coupled modified Boussinesq-like equations. Different nonlinear solutions of the coupled system are discussed, based on localized travelling wave ansätze and on coupled equations for the envelopes of co-propagating quasi-linear waves.

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I. INTRODUCTION

Recent studies of various collective processes in dust contaminated plasmas (DP) [1] have been of significant interest in relation with linear and nonlinear waves which are observed in laboratory and space plasmas. An issue of particular importance is the formation of strongly coupled DP crystals by highly charged dust grains, for instance in the sheath region above a horizontal negatively biased electrode in experiments [1, 2]. Low-frequency oscillations may occur in these mesoscopic dust grain quasi-lattices, in both longitudinal (acoustic mode) [3] and transverse (in-plane shear acoustic mode, off-plane optic-like mode) directions, as theoretically predicted and experimentally observed (see in Ref. [1] for a review).

In this paper, we focus on the nonlinear description of dust grain displacements in a one-dimensional dust crystal, which is suspended in a levitated horizontal equilibrium position where gravity and electric (or, possibly magnetic [4]) forces balance each other. Considering the coupling between the horizontal ($\sim \hat{x}$) and vertical (off-plane, $\sim \hat{z}$) degrees of freedom, and an arbitrary inter-grain interaction potential form $U(r)$ (e.g. Debye or else) and sheath potential $\Phi(z)$ (not necessary parabolic), we aim in deriving a set of equations which should serve as a basis for forthcoming studies of the nonlinear behaviour of longitudinal and transverse dust lattice waves (LDLWs, TDLWs) propagating in these crystals. The relation to recent studies of a similar scope (here recovered as special cases) is also discussed.

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II. THE MODEL

Let us consider a layer of charged dust grains (mass M and charge q, both assumed constant for simplicity) of lattice constant r_0. The Hamiltonian of such a chain is of the form

$$H = \sum_n \frac{1}{2} M \left(\frac{d\mathbf{r}_n}{dt} \right)^2 + \sum_{m \neq n} U(r_{nm}) + \Phi_{ext}(\mathbf{r}_n) \, ,$$

where $\mathbf{r}_n$ is the position vector of the $n-$th grain; $U_{nm}(r_{nm}) \equiv q\,\phi(x)$ is a binary interaction potential function related to the electrostatic potential $\phi(x)$ around the $m-$th grain, and $r_{nm} = |\mathbf{r}_n - \mathbf{r}_m|$ is the distance between the $n-$th and $m-$th grains. The external potential $\Phi_{ext}(\mathbf{r})$ accounts for the external force fields in which the crystal is embedded; in specific, Φ_{ext} takes into account the forces acting on the grains (and balancing each other at equilibrium, ensuring stability) in the vertical direction (i.e. gravity, electric and/or magnetic forces); it may also include the parabolic horizontal confinement potential imposed in experiments for stability [5] as well as, for completeness, the initial laser excitation triggering the oscillations in experiments.

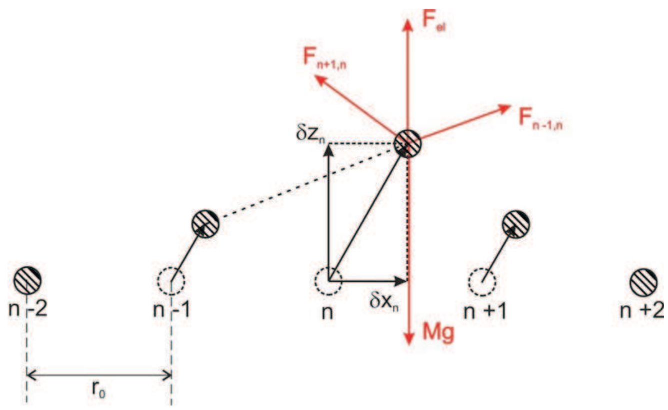

FIG. 1: Dust grain vibrations in the longitudinal ($\sim \hat{x}$) and transverse ($\sim \hat{z}$) directions, in a 1d dust lattice.

A. 2d equation of motion

Considering the motion of the $n-$th dust grain in both the *longitudinal* (horizontal, $\sim \hat{x}$) and the *transverse* (vertical, off–plane, $\sim \hat{z}$) directions (i.e. suppressing the transverse in-plane – shear – component, $\sim \hat{x}$), so that $\mathbf{r}_n = (x_n, z_n)$, we have the two-dimensional (in x, z) equation of motion

$$M \left(\frac{d^2\mathbf{r}_n}{dt^2} + \nu \frac{d\mathbf{r}_n}{dt} \right) = - \sum_n \frac{\partial U_{nm}(r_{nm})}{\partial \mathbf{r}_n} + \mathbf{F}_{ext}(\mathbf{r}_n) \equiv q\,\mathbf{E}(\mathbf{r}_n) + \mathbf{F}_{ext}(\mathbf{r}_n) \, , \tag{1}$$

where $E_j(x) = -\partial\phi(\mathbf{r})/\partial x_j$ is the (interaction) electrostatic field and $F_{ext,j} = -\partial\Phi_{ext}(x)/\partial x_j$ accounts for all external forces in the $j-$ direction ($j = 1/2$ for $x_j = x/z$); the usual *ad hoc* damping term was introduced in the left-hand-side of Eq. (1), involving the damping rate ν due to dust–neutral collisions.

B. Nonlinear vertical confining potential

We shall assume a smooth, continuous variation of the (generally inhomogeneous) field intensities $\mathbf{E}$ and/or $\mathbf{B}$, as well as the grain charge q (which may vary due to charging processes) near the equilibrium position $z_0 = 0$. Thus, we may develop

$$E(z) \approx E_0 + E_0' \, z + \frac{1}{2} E_0'' \, z^2 + \dots ,$$

$$B(z) \approx B_0 + B_0' \, z + \frac{1}{2} B_0'' \, z^2 + \dots ,$$

and

$$q(z) \approx q_0 + q_0' z + \frac{1}{2} q_0'' z^2 + \dots ,$$

where the prime denotes differentiation with respect to z and the subscript '0' denotes evaluation at $z = z_0$, viz. $E_0 = E(z = z_0)$, $E_0' = dE(z)/dz|_{z=z_0}$ and so forth. Accordingly, the electric force $F_e = q(z)E(z)$ and the magnetic force $F_m = -\partial(mB)/\partial z = -2\alpha\, B\, \partial B/\partial z$ (where the grain magnetic moment μ is related to the grain radius a and permeability μ via $m = (\mu - 1)a^3\, B/(\mu + 2) \equiv \alpha B$ [6]), which are now expressed as

$$F_e(z) \approx q_0 E_0 + (q_0 E_0' + q_0' E_0)\, z + \frac{1}{2}(q_0 E_0'' + 2 q_0' E_0' + q_0'' E_0)\, z^2 + \dots ,$$

and

$$F_m(z) \approx -2\alpha B_0 B_0' - 2\alpha({B_0'}^2 + B_0 {B''}_0)\, z - \alpha(B_0 B_0''' + 3 B_0' {B''}_0)\, z^2 + \dots ,$$

may be combined to give

$$F_e + F_m = -\frac{\partial \Phi}{\partial z} ,$$

where we have introduced the phenomenological potential $\Phi(z)$

$$\begin{aligned} \Phi(z) &\approx \Phi(z_0) + \left.\frac{\partial \Phi}{\partial z}\right|_{z=z_0} z + \frac{1}{2!} \left.\frac{\partial^2 \Phi}{\partial z^2}\right|_{z=z_0} z^2 + \frac{1}{3!} \left.\frac{\partial^3 \Phi}{\partial z^3}\right|_{z=z_0} z^3 + \dots \\ &\equiv \Phi_0 + \Phi_{(1)}\, z + \frac{1}{2}\, \Phi_{(2)}\, z^2 + \frac{1}{6}\, \Phi_{(3)}\, z^3 + \cdots . \end{aligned} \tag{2}$$

The definitions of $\Phi_{(j)} \equiv \left(\partial^j \Phi(z)/\partial z^j\right|_{z=z_0} = -(qE_0)_0^{(j-1)} + \alpha(B^2)_0^{(j)}$ (here, the superscript within parenthesis obviously denotes the order in partial differentiation; $j = 1, 2, \dots$) are obvious:

$$\begin{aligned} \Phi_{(1)} &= -(qE)_0 + \alpha(B^2)_0' = -q_0 E_0 + 2\alpha B_0 B_0' \\ \Phi_{(2)} &= -(qE_0)_0' + \alpha(B^2)_0'' \\ &= -(q_0' E_0 + q_0 E_0') + 2\alpha({B'}_0^2 + B_0 B_0'') \\ \Phi_{(3)} &= -(qE_0)_0'' + \alpha(B^2)_0''' \\ &= -(q_0'' E_0 + 2 q_0' E_0' + q_0 E_0'') + 2\alpha(3 B_0' B_0'' + B_0 B_0''') , \end{aligned} \tag{3}$$

and so forth. Obviously, $\Phi_{ext} = \Phi - Mgz$. The (vertical) force balance equation $\partial \Phi_{ext}/\partial z = 0$, viz.

$$Mg = q_0 E_0 - 2\alpha B_0 B_0' ,$$

is satisfied at equilibrium.

III. DISCRETE EQUATIONS OF MOTION

Assuming small displacements from equilibrium, one may Taylor expand the interaction potential energy $U(r)$ around the equilibrium inter-grain distance $l r_0 = |n - m| r_0$ (between $l-$th order neighbors, $l = 1, 2, \dots$), i.e. around $\delta x_n \approx 0$ and $\delta z_n \approx 0$, viz.

$$U(r_{nm}) = \sum_{l'=0}^{\infty} \frac{1}{l'!} \left.\frac{d^{l'} U(r)}{dr^{l'}}\right|_{r=l\,|n-m| r_0} (x_n - x_m)^{l'} ,$$

where l' denotes the degree of nonlinearity involved in its contribution: $l' = 1$ is the linear interaction term, $l' = 2$ stands for the quadratic potential nonlinearity, and so forth. Notice that the inter-grain distance $r = [(x_n - x_m)^2 + (z_n - z_m)^2]^{1/2}$ also needs to be expanded, i.e. near $|x_n - x_m| = l r_0$ and $z_n - z_m = 0$, so that $\partial U(r)/\partial x_j = (\partial U(r)/\partial r)(\partial r/\partial x_j) \approx \dots$. Obviously, $\delta x_n = x_n - x_n^{(0)}$ and $\delta z_n = z_n - z_n^{(0)}$ denotes the displacement of the $n-$th grain from the equilibrium position $(x_n^{(0)}, z_n^{(0)}) = (nr_0, 0)$. Retaining only nearest-neighbor interactions ($l = 1$), we obtain the coupled equations of motion

$$\begin{aligned}\frac{d^2(\delta x_n)}{dt^2} + \nu\,\frac{d(\delta x_n)}{dt} &= \omega_{0,L}^2\,(\delta x_{n+1} + \delta x_{n-1} - 2\delta x_n) \\ &\quad - a_{20}\left[(\delta x_{n+1} - \delta x_n)^2 - (\delta x_n - \delta x_{n-1})^2\right] \\ + a_{30}\left[(\delta x_{n+1} - \delta x_n)^3 - (\delta x_n - \delta x_{n-1})^3\right] &+ a_{02}\left[(\delta z_{n+1} - \delta z_n)^2 - (\delta z_n - \delta z_{n-1})^2\right] \\ &\quad - a_{12}\left[(\delta x_{n+1} - \delta x_n)(\delta z_{n+1} - \delta z_n)^2 - (\delta x_n - \delta x_{n-1})(\delta z_n - \delta z_{n-1})^2\right],\end{aligned} \tag{4}$$

and

$$\begin{aligned}\frac{d^2(\delta z_n)}{dt^2} + \nu\,\frac{d(\delta z_n)}{dt} &= \omega_{0,T}^2\,(2\delta z_n - \delta z_{n+1} + \delta z_{n-1}) - \omega_g^2\,\delta z_n \\ &\quad - K_1\,(\delta z_n)^2 - K_2\,(\delta z_n)^3 + \frac{a_{02}}{r_0}\left[(\delta z_{n+1} - \delta z_n)^3 - (\delta z_n - \delta z_{n-1})^3\right] \\ &\quad + 2\,a_{02}\left[(\delta x_{n+1} - \delta x_n)(\delta z_{n+1} - \delta z_n) - (\delta x_n - \delta x_{n-1})(\delta z_n - \delta z_{n-1})\right] \\ &\quad - a_{12}\left[(\delta x_{n+1} - \delta x_n)^2(\delta z_{n+1} - \delta z_n) - (\delta x_n - \delta x_{n-1})^2(\delta z_n - \delta z_{n-1})\right],\end{aligned} \tag{5}$$

where we have defined the longitudinal/transverse oscillation characteristic frequencies

$$\omega_{0,L}^2 = U''(r_0)/M\,, \qquad \omega_{0,T}^2 = -\,U'(r_0)/(Mr_0)\,, \tag{6}$$

(both assumed to be positive for any given form of interaction potential U) and the quantities

$$\begin{aligned}a_{20} = -\frac{1}{2M}\,U'''(r_0)\,, &\qquad a_{02} = -\frac{1}{2Mr_0^2}\left[U'(r_0) - r_0 U''(r_0)\right], \\ a_{30} = \frac{1}{6M}\,U''''(r_0)\,, &\qquad a_{12} = -\frac{1}{Mr_0^3}\left[U'(r_0) - r_0 U''(r_0) + r_0^2\,\frac{1}{2}\,U'''(r_0)\right],\end{aligned} \tag{7}$$

which are related to coupling nonlinearities. The *gap frequency* ω_g and the nonlinearity coefficients K_1 and K_2 are related to the form of the sheath environment (i.e. the potential Φ) via

$$\omega_g^2 = \Phi_{(2)}/M\,, \qquad K_1 = \Phi_{(3)}/(2M)\,, \qquad K_2 = \Phi_{(4)}/(6M)\,. \tag{8}$$

Obviously, the prime denotes differentiation, viz. $U''(r_0) = d^2U(r)/dr^2\big|_{r=r_0}$ and so on. In the above equations of motion, we have distinguished the linear contributions of the first neighbors from the nonlinear ones, i.e. the first line in the right–hand–side from the remaining ones, in both equations. Note that all of the coefficients are defined in such a way that they bear *positive* values for Debye–type interactions, i.e. if $U_D(r) = (q^2/r)\exp(-r/\lambda_D)$ (λ_D is the effective Debye length) since odd/even derivatives are then negative/positive; however, the sign of these coefficients is not a priori prescribed for a different interaction potential $U(r)$. Indeed, we insist on expressing all formulae in such a manner that a different interaction law may easily be assumed in a *"plug–in"* manner; in particular, even though the Debye potential U_D is widely accepted in DP crystal models, we think of the modification of U when one takes into account a magnetic field [4] or the ion flow towards the negative electrode [7]. Nevertheless, we provide the explicit form of the coefficients a_{ij} defined above for a Debye potential, for clarity, in the Appendix.

Upon careful inspection of the discrete equations of motion above, one notices that the lowest order nonlinearity in the longitudinal motion is due to the intergrain interaction law, while nonlinearity in the vertical motion is primarily induced by the coupling to the horizontal component (and, to less extent, by interactions).

IV. CONTINUUM APPROXIMATION

Adopting the standard continuum approximation, we may assume that only small displacement variations occur between neighboring sites, i.e.

$$\delta x_{n\pm1} \approx u \pm r_0 \frac{\partial u}{\partial x} + \frac{1}{2} r_0^2 \frac{\partial^2 u}{\partial x^2} \pm \frac{1}{3!} r_0^3 \frac{\partial^3 u}{\partial x^3} + \frac{1}{4!} r_0^4 \frac{\partial^4 u}{\partial x^4} \pm \dots,$$

where the (horizontal) displacement $\delta x_n(t)$ is now expressed via a continuous function $u = u(x,t)$. The analogous continuous function $w = w(x,t)$ is defined for $\delta z_n(t)$.

One may now proceed by inserting this ansatz in the discrete equations of motion (4, 5), and carefully evaluating the contribution of each term. The calculation, quite tedious yet perfectly straightforward, leads to a set of coupled continuum equations of motion in the form

$$\begin{aligned} \ddot{u} + \nu\, \dot{u} - c_L^2\, u_{xx} - \frac{c_L^2}{12}\, r_0^2\, u_{xxxx} \,&=\, -\,2\, a_{20}\, r_0^3\, u_x\, u_{xx} \,+\, 2\, a_{02}\, r_0^3\, w_x\, w_{xx} \\ &- a_{12}\, r_0^4 \left[(w_x)^2\, u_{xx} + 2 w_x w_{xx} u_x\right] \,+\, 3\, a_{30}\, r_0^4\, (u_x)^2\, u_{xx}\,, \end{aligned} \tag{9}$$

$$\begin{aligned} \ddot{w} + \nu\, \dot{w} + c_T^2\, w_{xx} + \frac{c_T^2}{12}\, r_0^2\, w_{xxxx} + \omega_g^2\, w \,&=\, -\, K_1\, w^2 \,-\, K_2\, w^3 \\ &+ 2\, a_{02}\, r_0^3\, (u_x\, w_{xx} \,+\, w_x\, u_{xx}) \\ + 3\, a_{02}\, r_0^3\, (w_x)^2\, w_{xx} \,&-\, a_{12}\, r_0^4 \left[(u_x)^2\, w_{xx} + 2 u_x u_{xx} w_x\right], \end{aligned} \tag{10}$$

where higher-order nonlinear terms were omitted. We have defined the characteristic velocities $c_L = \omega_{0,L}\, r_0$ and $c_T = \omega_{0,T}\, r_0$; the subscript x denotes partial differentiation, so that $u_x\, u_{xx} = (u_x^2)_x/2$ and $(u_x)^2\, u_{xx} = (u_x^3)_x/3$. Remember that the *gap frequency* ω_g and the coefficients K_1 and K_2 are related to the form of the sheath electric and/or magnetic potential via (8) above, viz. $F_{el} = M\, g - M\, \omega_g^2\, z - K_1\, z^2 - K_2\, z^3$.

V. RELATION TO PREVIOUS RESULTS - DISCUSSION

As a matter of fact, all known older results are based on equations which are readily recovered, as special cases, from Eqs. (4) and (5) and/or their continuum counterparts (9) and (10). In particular, the coupled Eqs. (1) and (2) in Ref. [8] are exactly recovered from (4) and (5), upon neglecting a_{30}, a_{12}, K_1 and K_2 and then evaluating all coefficients for a Debye–type potential.

Upon switching off the coupling (i.e. setting $w \to 0$), Eq. (9) above recovers exactly the nonlinear Eq. (13) in Ref. [10], which was therein shown to model (nonlinear) *longitudinal* dust grain motion in terms of (either Korteweg-de Vries– [3, 9] or Boussinesq–type) solitons; also see Eq. (2) in [11] (treating the formation of asymmetric envelope modulated LDLWs) and Eq. (2) in [12] (keep only first-neighbor interactions therein, to compare). In a similar manner, considering purely *transverse* motion (i.e. setting $u \to 0$) Eqs. (5) and (10) herein recover exactly the nonlinear Eqs. (7) and (8) in Ref. [13], where they were shown to model the amplitude modulation of TDLWs which is due to the sheath nonlinearity. Finally, needless to say, the linear limit recovers exactly the known equations of motion for either purely longitudinal or purely transverse motion (i.e. considering $a_{ij} = K_j = 0,\ \forall\, i, j$).

An exact treatment of the coupled evolution Eqs. (4, 5) – or, at least, the continuum system (9, 10) – seems quite a complex task to accomplish. Even though Eq. (9) may straightforward be seen as a Boussinesq–type equation [10], which is now modified by the coupling, its transverse counterpart (10) (for $u \to 0$, say) substantially differs from any known nonlinear model equation, bearing known exact solutions. Therefore, we shall limit ourselves to reporting this system of evolution equations, for the first time, thus keeping a more thorough investigation (analytical and/or numerical) of their fully nonlinear regime for a later report.

VI. COUPLED-MODE MODULATED WAVE PACKETS

In order to gain some insight regarding the influence of the mode–coupling on the nonlinear profile of the dust lattice waves, we may consider the effects which come into play when the amplitude of the LDLWs and the TDLWs – which are initially uncoupled in the small amplitude (linear) limit – is increased to a slightly finite (i.e. non negligible) value, thus allowing for a weak coupling between the two modes and a tractable appearance of the signature of the (weak) nonlinearity in the dynamics.

The standard way for such an approach is via the introduction of multiple space and time scales, viz. $X_0, X_1, X_2, ...$ and $T_0, T_1, T_2, ...$, where $X_n = \epsilon^n x$ and $T_n = \epsilon^n t$ ($\epsilon \ll 1$ is a smallness parameter). The solutions are expanded as: $u = \epsilon u_1 + \epsilon^2 u_2 + ...$ (plus an analogous expression for w). The technical details of the calculation are described e.g. in [11] and will be omitted here. We shall apply this reductive perturbation technique to the system obtained from Eqs. (9, 10) by keeping only the lowest-order nonlinear terms (i.e. omitting the last line in both equations); we set $p_0 = 2a_{20}r_0^3$ and $h_0 = 2a_{02}r_0^3$ for simplicity. Note the inevitable (and qualitatively expected) complication of the calculation due to the different dispersion laws in the two modes [14].

The first-order ($\sim \epsilon$) equations are uncoupled and may be solved by assuming $\{u_1, w_1\} = \{\psi_L^{(0)}, \psi_T^{(0)}\} + [\{\psi_L, \psi_T\} \exp i(kx - \omega t) + \text{c.c.}]$ (complex conjugate). Upon substitution, we obtain $\psi_T^{(0)} = 0$; the remaining (3) amplitudes are left arbitrary. This readily yields the known dispersion relations

$$\omega_L^2 + i\nu\omega_L = c_L^2 k^2 \left(1 - \frac{k^2 r_0^2}{12}\right), \qquad \omega_T^2 + i\nu\omega_T = \omega_g^2 - c_T^2 k^2 \left(1 - \frac{k^2 r_0^2}{12}\right), \tag{11}$$

for the (acoustic) LDL and the (optical-like) TDL mode respectively.

The 2nd-order ($\sim \epsilon^2$) equations contain secular (1st-harmonic forcing) terms, whose elimination imposes a pair of conditions in the form: $\partial\Psi_j/\partial T_1 + v_{g,j}\, \partial\Psi_j/\partial X_1 = 0$ (where $j \in \{1, 2\} \equiv \{L, T\}$ in the following), implying that the amplitudes Ψ_j travel at the (different) group velocities $v_{g,j} \equiv \partial\omega_j(k)/\partial k$. See that $v_{g,T} = \omega_T{}'(k) < 0$ (the TDLW is a *backward wave*), as immediately obtained from (11b). The remaining system is then solved for the 0th and the 2nd harmonic amplitudes (in ϵ^2) [14]; the solution finally obtained is of the form:

$$\begin{aligned}
\delta x_n(t) &\approx u(x,t) \approx \epsilon[\psi_0 + \psi_1 \exp i(kx - \omega_1 t) + \text{c.c.}] + \epsilon^2\, u_2^{(2)} \exp 2i(kx - \omega_1 t) + \text{c.c.}] + \mathcal{O}(\epsilon^3) \\
\delta z_n(t) &\approx w(x,t) \approx \epsilon[\psi_2 \exp i(kx - \omega_2 t) + \text{c.c.}] + \epsilon^2\, \{w_2^{(0)} + [w_2^{(2)} \exp 2i(kx - \omega_2 t) + \text{c.c.}]\} + \mathcal{O}(\epsilon^3)\,.
\end{aligned} \tag{12}$$

We henceforth denote the significant amplitudes $u_1^{(0)}$, $u_1^{(1)}$ and $w_1^{(1)}$ by Ψ_0, Ψ_1 and Ψ_2 respectively. The 2nd order correction amplitudes are

$$u_2^{(2)} = ik^3\, \frac{p_0\Psi_1^2 - h_0\Psi_2^2}{D_2^{(L)}}, \qquad w_2^{(0)} = -\frac{2K_1}{\omega_g^2}|\Psi_1|^2, \qquad w_2^{(2)} = -\frac{1}{D_2^{(T)}}\left(K_1\Psi_2^2 + 2ih_0k^3\Psi_1\Psi_2\right), \tag{13}$$

where $D_2^{(L)} = -c_L^2 r_0^2 k^4 + 2i\nu\omega_L$ and $D_2^{(T)} = -3\omega_g^2 + c_T^2 r_0^2 k^4 + 2i\nu\omega_T$. The contributions $u_2^{(1)}$, $w_2^{(1)}$ and $u_2^{(0)}$ are left arbitrary by the algebra and were thus set to zero.

Proceeding to the 3rd-order ($\sim \epsilon^3$) equations, the elimination of the secular terms together with zeroth order equations provide three explicit conditions, for $\Psi_{0,1,2}$. After some tedious algebra, these take the form

$$\begin{aligned}
&i\left(\frac{\partial\Psi_1}{\partial T_2} + v_{g,1}\frac{\partial\Psi_1}{\partial X_2}\right) + P_1\,\frac{\partial^2\Psi_1}{\partial X_1^2} + Q_{11}\,|\Psi_1|^2\Psi_1 + Q_{12}\,|\Psi_2|^2\Psi_1 + (Q_{0,1}\Psi_1 + Q_{0,2}\Psi_2)\frac{\partial\Psi_0}{\partial X_1} + H_1 = 0 \\
&i\left(\frac{\partial\Psi_2}{\partial T_2} + v_{g,2}\frac{\partial\Psi_2}{\partial X_2}\right) + P_2\,\frac{\partial^2\Psi_2}{\partial X_1^2} + Q_{22}\,|\Psi_2|^2\Psi_2 + Q_{21}\,|\Psi_1|^2\Psi_2 + H_2 = 0 \\
&\qquad (v_{g,1}^2 - c_L^2)\frac{\partial\Psi_0}{\partial X_1} = -p_0k^2|\Psi_1|^2 + h_0k^2|\Psi_2|^2 + C\,,
\end{aligned} \tag{14}$$

where C is an integration constant (to be determined by the boundary conditions). The linear dispersion terms P_j are related to the (curvature of) the dispersion relations (11) as $P_j = \omega_j{}''(k)$ ($j = 1, 2$); the group velocities $v_{g,j}$ were defined above [15]. The nonlinearity coefficients Q_{ij} ($i = 0, 1, 2$, $j = 1, 2$) and the 'peculiar' contributions H_j (involving cross-terms in $\Psi_i^2\Psi_j^*$) are too lengthy to report here [14]. Observe that, once C is determined, one may cast Eqs. (14) into the form of a (modified, asymmetric) system of coupled nonlinear Schrödinger equations (CNLSE). Note that we have avoided the usual envelope (Galilean) transformation $\{x, t\} \to \{x - v_{g,j}t, t\}$, since it does not simplify this (asymmetric, with respect to $1 \leftrightarrow 2$) system. Finally, let us point out, for rigor, that the results in [11] and [16] are exactly recovered, from both (13) and (14), in the appropriate – uncoupled mode – limits (namely, $\Psi_2 \to 0$ and $\Psi_1 \to 0$, respectively, for LDLWs and TDLWs).

Despite the obvious analytical complication, the physical mechanism underlying the above results is rather transparent. There is an energy pumping effect between the zeroth-harmonic longitudinal (displacement) mode Ψ_0, first put forward in [11] (for LDLWs, yet long known in solid state physics [17]) and the modulated (low-frequency) LDL and (high-frequency) TDL mode(s). Note the strong misfit (asymmetry) between the dispersion laws dominating the coupled modes, despite which – regretfully – no simplifying assumption may be analytically carried out in this continuum model.

VII. CONCLUSION

We have put forward a comprehensive nonlinear model for coupled longitudinal-to-transverse displacements in a horizontal dust mono-layer, levitated in a sheath under the influence of gravity and an electric and/or magnetic field. All of the above results are generic, i.e. valid for any assumed form of the inter-grain interaction potential $U(r)$ and the sheath potential Φ, and will hopefully contribute to the elucidation of the grain oscillatory dynamics in dust crystals.

Appendix: Form of the coefficients for the Debye interaction potential

Consider the Debye potential (energy) $U_D(r) = q\phi_D(r) = q^2\, e^{-r/\lambda_D}/r$. Defining the (positive real) lattice parameter $\kappa = r_0/\lambda_D$, one straightforward has

$$U_D'(r_0) = -\frac{q^2}{\lambda_D^2}\, e^{-\kappa}\, \frac{1+\kappa}{\kappa^2}\,, \qquad U_D''(r_0) = +\frac{2q^2}{\lambda_D^3}\, e^{-\kappa}\, \frac{1+\kappa+\frac{\kappa^2}{2}}{\kappa^3}\,,$$

$$U_D'''(r_0) = -\frac{6q^2}{\lambda_D^4}\, e^{-\kappa}\, \frac{1+\kappa+\frac{\kappa^2}{2}+\frac{\kappa^3}{6}}{\kappa^4}\,, \qquad U_D''''(r_0) = +\frac{24q^2}{\lambda_D^5}\, e^{-\kappa}\, \frac{1+\kappa+\frac{\kappa^2}{2}+\frac{\kappa^3}{6}+\frac{\kappa^4}{24}}{\kappa^5}\,,$$

where the prime denotes differentiation and $l = 1, 2, 3, ...$ is a positive integer. Now, combining with definitions (6, 7), we have:

$$\omega_{L,0}^2 = \frac{2q^2}{M\lambda_D^3}\, e^{-\kappa}\, \frac{1+\kappa+\kappa^2/2}{\kappa^3} \equiv c_L^2/(\kappa^2\lambda_D^2)\ , \qquad \omega_{T,0}^2 = \frac{q^2}{M\lambda_D^3}\, e^{-\kappa}\, \frac{1+\kappa}{\kappa^3} \equiv c_T^2/(\kappa^2\lambda_D^2)\ ,$$

$$p_0 \equiv 2a_{20}\kappa^3\lambda_D^3 = \frac{6q^2}{M\lambda_D}\, e^{-\kappa}\left(\frac{1}{\kappa}+1+\frac{\kappa}{2}+\frac{\kappa^2}{6}\right)\,, \qquad h_0 \equiv 2a_{02}\kappa^3\lambda_D^3 = \frac{3q^2}{M\lambda_D}\, e^{-\kappa}\left(\frac{1}{\kappa}+1+\frac{\kappa}{3}\,,\right)$$

$$a_{30} = \frac{q^2}{6M\lambda_D^5}\, e^{-\kappa}\, \frac{1}{\kappa^5}\left(\kappa^4+4\kappa^3+12\kappa^2+24\kappa+24\right)\,, \qquad a_{12} = \frac{q^2}{2M\lambda_D^5}\, e^{-\kappa}\, \frac{1}{\kappa^5}\left(\kappa^3+5\kappa^2+12\kappa+12\right)\,.$$

Of course, all known previous definitions of (some of) these coefficients (for nearest neighbour interactions; see in the references cited in the text) are exactly recovered. Note, finally, that κ is of the order of (or slightly higher than) unity in experiments; therefore, all coefficients turn out to be of similar order of magnitude, as one may check numerically.

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[14] The tedious details are left to be reported in a lengthier report, in preparation.
[15] See that, once damping is taken into account (via $\nu \neq 0$), all dispersion-related quantities – namely ω_j, $v_{g,j}$ and P_j ($j = 1, 2 \equiv L, T$) here – bear an imaginary part. All of the corresponding expressions – readily obtained from Eq. (11) and thus omitted here – were exactly recovered by the (tedious) algebra within our perturbative scheme; recall, in passing, that $\Psi_{0,1,2}$ are complex numbers, by definition.
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