Lagrangean formulation of ion- and dust-ion-acoustic waves *

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Ion-acoustic modes propagating in unmagnetized dusty plasmas are studied by applying a generic collisionless fluid model. An Eulerian-to-Lagrangean variable transformation leads to a new system of evolution equations, which may be combined into a single (rather complex) equation for the mean ion density. The linear and weakly nonlinear oscillations regimes are studied and their modulational stability is investigated. This study extends known previous results on the Lagrangian description of electron- or ion-acoustic electrostatic plasma modes. The relevance with the description of dust-ion acoustic waves propagating in a dusty plasma is also discussed.

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I. INTRODUCTION

Dusty plasmas (DP) have recently attracted a great deal of attention due to a variety of new phenomena observed in them and the novel physical mechanisms involved in their description [1, 2]. In addition to known plasma electrostatic waves [3], e.g. ion-acoustic waves (IAW), new oscillatory modes arise in DP [1, 2], among which the *dust-ion acoustic wave* (DIAW), essentially corresponding to ion-acoustic (IA) oscillations in a strong presence of dust.

The linear properties of the IAWs and DIAWs have been quite extensively studied and now appear well understood [1]. As far as nonlinear effects are concerned, various studies have pointed out the possibility of the formation of DIAW-related localized structures, due to a mutual compensation between nonlinearity and dispersion, including small-amplitude pulse solitons, shocks and vortices [4]. Furthermore, the propagation of nonlinearly modulated DIA wave packets was studied in Ref. [5], in addition to the formation of localized envelope soliton–modulated waves due to the modulational instability of the carrier waves. A very interesting known approach is the Lagrangean description of a nonlinear wave profile. In the context of electrostatic plasma waves, this formalism has been employed in studies of electron plasma waves [6–8] and, more recently, ion-acoustic [9] and dust-acoustic [10] waves. Our aim here is to extend previous results by applying the Lagrangean formalism to the description of nonlinear DIAWs propagating in dusty plasmas.

II. THE MODEL

We shall consider the nonlinear propagation of dust-ion-acoustic waves in a collisionless plasma consisting of three distinct particle species 'α': an inertial species of ions (denoted by 'i'; mass m_i, charge $q_i = +Z_i e$; e denotes the absolute of the electron charge), surrounded by an environment of thermalized electrons (mass m_e, charge $-e$), and massive dust grains (mass M, charge $q_d = sZ_d e$, both assumed constant for simplicity); Z_d denotes the charge state of dust grains; we leave the choice of dust grain charge sign $s = q_d/|q_d|$ ($= -1/+1$ for negative/positive dust charge) open in the algebra. Charge neutrality is assumed at equilibrium.

The hydrodynamic–Poisson system of equations which describe the evolution of the ion 'fluid' in the plasma are considered. The ion number density n_i is governed by the continuity equation

$$\frac{\partial n_i}{\partial t} + \nabla(n_i \, \mathbf{u}_i) = 0 \, , \tag{1}$$

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where the mean velocity $\mathbf{u}_i$ obeys

$$\frac{\partial \mathbf{u}_i}{\partial t} + \mathbf{u}_i \cdot \nabla \mathbf{u}_i \, = \, \frac{Z_i e}{m_i} \mathbf{E} \, = \, - \frac{Z_i e}{m_i} \nabla \Phi \, . \tag{2}$$

The electric field $\mathbf{E} = -\nabla \Phi$ is related to the gradient of the wave potential Φ, which is obtained from Poisson's equation $\nabla \cdot \mathbf{E} = 4\pi \sum q_s \, n_s$, viz.

$$\nabla^2 \Phi \, = \, 4\pi \, e \, (n_c \, + n_h \, - Z_i \, n_i) \, . \tag{3}$$

Alternatively, one may consider

$$\frac{\partial E}{\partial t} \, = \, -4\pi \sum_\alpha q_\alpha \, n_\alpha \, u_\alpha \, . \tag{4}$$

We assume a near-Boltzmann distribution for the electrons, i.e. $n_e \approx n_{e,0} \, \exp(e\Phi/k_B T_e)$ (T_e is the electron temperature and k_B is Boltzmann's constant). The dust distribution is assumed stationary, i.e. $n_d \approx$ const.. The overall quasi-neutrality condition at equilibrium then reads

$$Z_i \, n_{i,0} \, + s Z_d \, n_d \, - n_{e,0} \, = \, 0 \, . \tag{5}$$

A. Reduced Eulerian equations

By choosing appropriate physical scales, Eqs. (1)-(3) can be cast into a reduced (dimensionless) form. Let us define the ion-acoustic speed $c_s = (k_B T_e/m_i)^{1/2}$. An appropriate choice for the space and timescales, L and $T = L/c_s$, are the effective Debye length $\lambda_D = (k_B T_e/4\pi Z_i^2 n_{i,0} e^2)^{1/2} \equiv c_s/\omega_{p,i}$ and the ion plasma period $\omega_{p,i}^{-1} = (4\pi n_{i,0} Z_i^2 e^2/m_i)^{-1/2}$, respectively. Alternatively, one might leave the choice of L (and thus $T = L/c_s$) arbitrary – following an idea suggested in Refs. [9, 10]) – which leads to the appearance of a dimensionless dispersion parameter $\delta = 1/(\omega_{p,i} T) = \lambda_D/L$ in the formulae. The specific choice of scale made above corresponds to $\delta = 1$ (implied everywhere in the following, unless otherwise stated); however, we may keep the parameter δ to 'label' the dispersion term in the forthcoming formulae.

For one-dimensional wave propagation along the x axis, Eqs. (1) - (3) can now be written as

$$\begin{aligned} \frac{\partial n}{\partial t} + \frac{\partial (n \, u)}{\partial x} \, &= \, 0 \, , \\ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \, &= \, -\nabla \phi \, , \\ \delta^2 \, \frac{\partial^2 \phi}{\partial x^2} \, &= \, (\hat{n} - n) \, , \end{aligned} \tag{6}$$

where all quantities are dimensionless: $n = n_i/n_{i,0}$, $\mathbf{u} = \mathbf{u}_i/v_0$ and $\phi = \Phi/\Phi_0$; the scaling quantities are, respectively: the equilibrium ion density $n_{i,0}$, the effective sound speed $v_0 = c_s$ (defined above) and $\Phi_0 = k_B T_e/(Z_i e)$. The (reduced) electron and dust background density $\hat{n}$ is defined as

$$\hat{n} = \frac{n_e}{Z_i n_{i,0}} \, e^{\phi/Z_i} + s \frac{Z_d n_d}{Z_i n_{i,0}} \, \equiv \mu \, e^{\phi/Z_i} + 1 - \mu \, , \tag{7}$$

where we have defined the DP parameter $\mu = n_{e,0}/(Z_i n_{i,0})$, and made use of Eq. (5). Note that both n and $\hat{n}$ reduce to unity at equilibrium.

We shall define, for later reference, the function $f(\phi) = \hat{n}$ – given by Eq. (7) – and its inverse function

$$f^{-1}(x) = Z_i \, \ln\left(1 + \frac{x-1}{\mu}\right) \equiv g(x) \, , \tag{8}$$

viz. $f(\phi) = x$ implies $\phi = f^{-1}(x) \equiv g(x)$.

We note that the dependence on the charge sign s is now incorporated in $\mu = 1 + s Z_d n_{d,0}/(Z_i n_{i,0})$; retain that $\mu < 1$ ($\mu > 1$) corresponds to negative (positive) dust. Remarkably, since the dust-free limit is recovered for $\mu = 1$, the results to be obtained in the following are also straightforward valid for ion-acoustic waves propagating in (dust-free) e-i plasma, upon setting $\mu = 1$ in the formulae.

The well–known DIAW dispersion relation $\omega^2 = c_s^2 k^2/(k^2\lambda_D^2 + 1)$ [5] is obtained from Eqs. (1) to (5). On the other hand, the system (6) yields the reduced relation $\omega^2 = k^2/(\delta^2 k^2 + 1)$, which of course immediately recovers the former dispersion relation upon restoring dimensions (regardless, in fact, of one's choice of space scale L; cf. definition of δ). However, some extra qualitative information is admittedly hidden in the latter (dimensionless) relation. Should one consider a very long space scale $L \gg \lambda_D$ (i.e. $\delta \ll 1$), one readily obtains $\omega \sim k$ (unveiling the role of δ as a characteristic dispersion control parameter). Finally, the opposite limit of short L (or infinite δ) corresponds to ion plasma oscillations (viz. $\omega = \omega_{p,i} =$ constant).

B. Lagrangean description

Let us introduce the Lagrangean variables $\{\xi, \tau\}$, which are related to the Eulerian ones $\{x, t\}$ via

$$\xi = x - \int_0^\tau u(\xi, \tau')\, d\tau' , \qquad \tau = t . \tag{9}$$

See that they coincide at $t = 0$. Accordingly, the space and time gradients are transformed as

$$\partial/\partial x \to \alpha^{-1}\, \partial/\partial\xi , \qquad \partial/\partial t \to \partial/\partial\tau - \alpha^{-1}\, u\, \partial/\partial\xi ,$$

where we have defined the quantity

$$\alpha(\xi, \tau) \equiv \frac{\partial x}{\partial \xi} = 1 + \int_0^\tau d\tau' \frac{\partial}{\partial\xi} u(\xi, \tau') . \tag{10}$$

Note that the convective derivative $D \equiv \partial/\partial t + u\,\partial/\partial x$ is now plainly identified to $\partial/\partial\tau$. Also notice that α satisfies $\alpha(\xi, \tau = 0) = 0$ and

$$\frac{\partial\alpha(\xi,\tau)}{\partial\tau} = \frac{\partial u(\xi,\tau)}{\partial\xi} \tag{11}$$

As a matter of fact, the Lagrangean transformation defined here reduces to a Galilean transformation if one suppresses the evolution of u, i.e. for $u =$ const. (or $\partial u/\partial\tau = \partial u/\partial\xi = 0$, hence $\alpha = 1$). Furthermore, if one also suppresses the dependence in time τ, this transformation is reminiscent of the travelling wave ansatz $f(x, t) = f(x - vt \equiv s)$, which is widely used in the Sagdeev potential formalism [4, 11].

The Lagrangean variable transformation defined above leads to a new set of reduced equations

$$n(\xi,\tau) = \alpha^{-1}(\xi,\tau)\, n(\xi, 0) \tag{12}$$

$$\frac{\partial u(\xi,\tau)}{\partial\tau} = \frac{Z_i e}{m_i} E(\xi,\tau) = -\frac{Z_i e}{m_i}\, \alpha^{-1}(\xi,\tau)\, \frac{\partial\phi(\xi,\tau)}{\partial\xi} \tag{13}$$

$$\alpha^{-1}(\xi,\tau)\, \frac{\partial E(\xi,\tau)}{\partial\xi} = 4\pi Z_i e[n(\xi,\tau) - \hat{n}\, n_{i,0}] \tag{14}$$

$$\left(\frac{\partial}{\partial\tau} - \alpha^{-1} u\, \frac{\partial}{\partial\xi}\right) E(\xi,\tau) = -4\pi Z_i e n(\xi,\tau) u(\xi,\tau) , \tag{15}$$

where we have temporarily restored dimensions for physical transparency; recall that the (dimensionless) quantity $\hat{n}$, which is in fact a function of ϕ, is given by (7). One immediately recognizes the role of the (inverse of the) function $\alpha(\xi, \tau)$ as a density time evolution operator; cf. Eq. (12) [12]. Poisson's equation is now obtained by eliminating ϕ from Eqs. (13, 14)

$$\alpha^{-1}\, \frac{\partial}{\partial\xi}\left(\alpha^{-1}\, \frac{\partial\phi}{\partial\xi}\right) = -4\pi Z_i e(n - \hat{n}\, n_{i,0}) . \tag{16}$$

Note that a factor δ^2 should appear in the left-hand side if one rescaled Eq. (16) as described above; cf. the last of Eqs. (6). This will be retained for later reference, with respect to the treatment suggested in Ref. [9] (see discussion below).

In principle, our aim is to solve the system of Eqs. (12) to (15) or, by eliminating ϕ, Eqs. (12), (13) and (16) for a given initial condition $n(\xi, \tau = 0) = n_0(\xi)$, and then make use of the definition (9) in order to invert back to the Eulerian arguments of the state moment variables (i.e. density, velocity etc.). However, this abstract scheme is definitely not a trivial task to accomplish.

III. NONLINEAR DUST-ION ACOUSTIC OSCILLATIONS

Multiplying Eq. (14) by $u(\xi,\tau)$ and then adding to Eq. (15), one obtains

$$\frac{\partial E(\xi,\tau)}{\partial \tau} = -4\pi Z_i e n_{i,0}\, \hat{n}\, u(\xi,\tau)\,. \tag{17}$$

Combining with Eq. (13), one obtains

$$\frac{\partial^2 u}{\partial \tau^2} = -\omega_{p,i}^2\, \hat{n}\, u\,, \tag{18}$$

where $\omega_{p,i}$ is the ion plasma frequency (defined above). Despite its apparent simplicity, Eq. (18) is *neither* an ordinary differential equation (ODE) – since all variables depend on *both* time τ and space ξ – *nor* a closed evolution equation for the mean velocity $u(\xi,\tau)$: note that the (normalized) background particle density $\hat{n}$ depends on the potential ϕ and on the plasma parameters; see its definition (7). The evolution of the potential $\phi(\xi,\tau)$, in turn, involves $u(\xi,\tau)$ (via the quantity $\alpha(\xi,\tau)$) and the ion density $n(\xi,\tau)$.

Eq. (18) suggests that the system performs nonlinear oscillations at a frequency $\omega = \omega_{p,i}\, \hat{n}^{1/2}$. Near equilibrium, the quantity $\hat{n}$ is approximately equal to unity and one plainly recovers a linear oscillation at the ion plasma frequency $\omega_{p,i}$. Quite unfortunately this apparent simplicity, which might in principle enable one to solve for $u(\xi,\tau)$ and then obtain $\{\xi,\tau\}$ in terms of $\{x,t\}$ and vice versa (cf. Davidson's treatment for electron plasma oscillations in Ref. [7]; also compare to Ref. [8], setting $\gamma = 0$ therein), is absent in the general (off-equilibrium) case where the plasma oscillations described by Eq. (18) are intrinsically *nonlinear*.

Since Eq. (18) is in general not a closed equation for u, unless the background density $\hat{n}$ is constant (i.e. independent of ϕ, as in Refs. [7, 8]), one can *neither* apply standard methods involved in the description of nonlinear oscillators on Eq. (18) (cf. Ref. [8]), *nor* reduce the description to a study of Eqs. (18, 17) (cf. Ref. [6]), but rather has to retain all (or rather five) of the evolution equations derived above, since five inter-dependent dynamical state variables (i.e. n, u, E, ϕ and α) are involved. This procedure will be exposed in the following Section.

IV. PERTURBATIVE NONLINEAR LAGRANGEAN TREATMENT

Let us consider weakly nonlinear oscillations performed by our system close to (but not at) equilibrium. The basis of our study will be the reduced system of equations

$$\begin{aligned}
\frac{\partial}{\partial \tau}(\alpha\, n) &= 0\,,\\
\frac{\partial u}{\partial \tau} &= E\,,\\
\frac{\partial E}{\partial \xi} &= (n-\hat{n})\,\alpha\,,\\
\alpha\, E &= -\frac{\partial \phi}{\partial \xi}\,,\\
\frac{\partial \alpha}{\partial \tau} &= \frac{\partial u}{\partial \xi}\,,
\end{aligned} \tag{19}$$

which follow from the Lagrangean Eqs. (12) to (16) by scaling over appropriate quantities, as described in §II A [13]. This system describes the evolution of the state vector, say $\mathbf{S} = (\alpha, n, u, E, \phi)$ $(\in \Re^5)$, in the Lagrangean coordinates defined above. We will consider small deviations from the equilibrium state $\mathbf{S}_0 = (1,1,0,0,0)^T$, by taking $\mathbf{S} = \mathbf{S}^{(0)} + \epsilon \mathbf{S}_1{}^{(0)} + \epsilon^2 \mathbf{S}_2{}^{(0)} + ...$, where ϵ $(\ll 1)$ is a smallness parameter. Accordingly, we shall Taylor develop the quantity $\hat{n}(\phi)$ near $\phi \approx 0$, viz. $\phi \approx \epsilon\phi_1 + \epsilon^2\phi_2 + ...$, in order to express $\hat{n}$ as

$$\begin{aligned}
\hat{n} &\approx 1 + c_1\phi + c_2\phi^2 + c_3\phi^3 + ...\\
&= 1 + \epsilon c_1\phi_1 + \epsilon^2(c_1\phi_2 + c_2\phi_1^2) + \epsilon^3(c_1\phi_3 + 2c_2\phi_1\phi_2 + c_3\phi_1^3) + ...\,,
\end{aligned} \tag{20}$$

where the coefficients c_j $(j = 1, 2, ...)$, which are determined from the definition (7) of $\hat{n}$, contain all the essential dependence on the plasma parameters, e.g. μ; making use of $e^x \approx \sum_{n=0}^{\infty} x^n/n!$, one readily obtains

$$c_1 = \mu/Z_i\,, \qquad c_2 = \mu/(2Z_i^2)\,, \qquad c_2 = \mu/(6Z_i^3)\,.$$

Remember that for $\mu = 1$ (i.e. for vanishing dust) one recovers the expressions for IAWs in e-i plasma.

Following the standard reductive perturbation technique [14], we shall consider the stretched (slow) Lagrangean coordinates $Z = \epsilon(\xi - \lambda\,\tau)$, $T = \epsilon^2\,\tau$ (where $\lambda \in \Re$ will be determined later). The perturbed state of (the j–th — $j = 1, ..., 5$ — component of) the state vector $\mathbf{S}^{(n)}$ is assumed to depend on the fast scales via the carrier phase $\theta = k\xi - \omega\tau$, while the slow scales enter the argument of the (j–th element's) l–th harmonic amplitude $S_{j,l}^{(n)}$, viz. $Sj^{(n)} = \sum_{l=-\infty}^{\infty} S_{j,l}^{(n)}(Z,T)\,e^{il(k\xi-\omega\tau)}$ (where $S_{j,-l}^{(n)} = {S_{j,l}^{(n)}}^*$ ensures reality). Treating the derivative operators as

$$\frac{\partial}{\partial\tau} \to \frac{\partial}{\partial\tau} - \epsilon\,\lambda\,\frac{\partial}{\partial Z} + \epsilon^2\,\frac{\partial}{\partial T}\,, \qquad \frac{\partial}{\partial\xi} \to \frac{\partial}{\partial\xi} + \epsilon\,\frac{\partial}{\partial Z}\,,$$

and substituting into the system of evolution equations, one obtains an infinite series in both (perturbation order) ϵ^n and (phase harmonic) l. The standard perturbation procedure now consists in solving in successive orders $\sim \epsilon^n$ and substituting in subsequent orders. The method involves a tedious calculation which is however straightforward; the details of the method are presented e.g. in Ref. [5], so only the essential stepstones need to be provided here.

The equations obtained for $n = l = 1$ determine the first harmonics of the perturbation

$$n_1^{(1)} = -\alpha_1^{(1)} = (k^2/\omega^2)\psi\,, u_1^{(1)} = (k/\omega)\psi\,, \qquad E_1^{(1)} = -ik\psi \tag{21}$$

where ψ denotes the potential correction $\phi_1^{(1)}$. The cyclic frequency ω obeys the dispersion relation $\omega^2 = k^2/(k^2 + sc_1)$, which exactly recovers, once dimensions are restored, the standard IAW dispersion relation [3] mentioned above.

Proceeding in the same manner, we obtain the second order quantities, namely the amplitudes of the second harmonics $\mathbf{S}_2^{(2)}$ and constant ('direct current') terms $\mathbf{S}_0^{(2)}$, as well as a finite contribution $\mathbf{S}_1^{(2)}$ to the first harmonics; as expected from similar studies, these three (sets of 5, at each n, l) quantities are found to be proportional to ψ^2, $|\psi|^2$ and $\partial\psi/\partial Z$ respectively; the lengthy expressions are omitted here for brevity. The ($n = 2$, $l = 1$) equations provide the compatibility condition: $\lambda = \omega(1-\omega^2)/k = d\omega/dk$; λ is therefore the group velocity $v_g(k) = \omega'(k)$ at which the wave envelope propagates. It turns out that v_g decreases with increasing wave number k; nevertheless, it always remains positive.

In order $\sim \epsilon^3$, the equations for $l = 1$ yield an explicit compatibility condition in the form of a nonlinear Schrödinger–type equation (NLSE)

$$i\,\frac{\partial\psi}{\partial T} + P\,\frac{\partial^2\psi}{\partial Z^2} + Q\,|\psi|^2\,\psi = 0\,. \tag{22}$$

Recall that $\psi \equiv \phi_1^{(1)}$ denotes the amplitude of the first-order electric potential perturbation. The 'slow' variables $\{Z, T\}$ were defined above.

The *dispersion coefficient* P is related to the curvature of the dispersion curve as $P = \omega''(k)/2 = -3\omega^3(1-\omega^2)/(2k^2)$. One may easily check that P is negative (for all values of k).

The *nonlinearity coefficient* Q is due to carrier wave self-interaction. It is given by the expression

$$Q = +\frac{\omega^3}{12\,k^4}\,\frac{\mu}{Z_i^4}\left[3Z_i^3k^6 - 3(\mu+4)Z_i^2k^4 + 3(1-2\mu-5\mu^2)Z_ik^2 - \mu(3\mu-1)^2\right]. \tag{23}$$

where the coefficients $c_{1,2,3}$ were defined above.

For low wavenumber k, Q goes to $-\infty$ as

$$Q \approx -\frac{(3\mu-1)^2\mu^{1/2}}{12\,Z_i^{5/2}}\,\frac{1}{k}\,.$$

A. Modulational stability analysis

According to the standard analysis [15], we can linearize around the plane wave solution of the NLSE (22) $\psi = \hat\psi\,e^{iQ|\hat\psi|^2\tau} + c.c.$, (*c.c.*: complex conjugate) – notice the amplitude dependence of the frequency shift $\Delta\omega = \epsilon^2 Q|\hat\psi|^2$ – by setting $\hat\psi = \hat\psi_0 + \epsilon\,\hat\psi_1$, and then assuming the perturbation $\hat\psi_1$ to be of the form: $\hat\psi_1 = \hat\psi_{1,0}\,e^{i(\hat k\zeta - \hat\omega\tau)} + c.c.$. Substituting into (22), one thus readily obtains $\hat\omega^2 = P^2\,\hat k^2\left(\hat k^2 - 2(Q/P)|\hat\psi_{1,0}|^2\right)$. The wave will thus be *stable* ($\forall\,\hat k$) if the product PQ is negative. However, for positive $PQ > 0$, instability sets in for wavenumbers below a critical

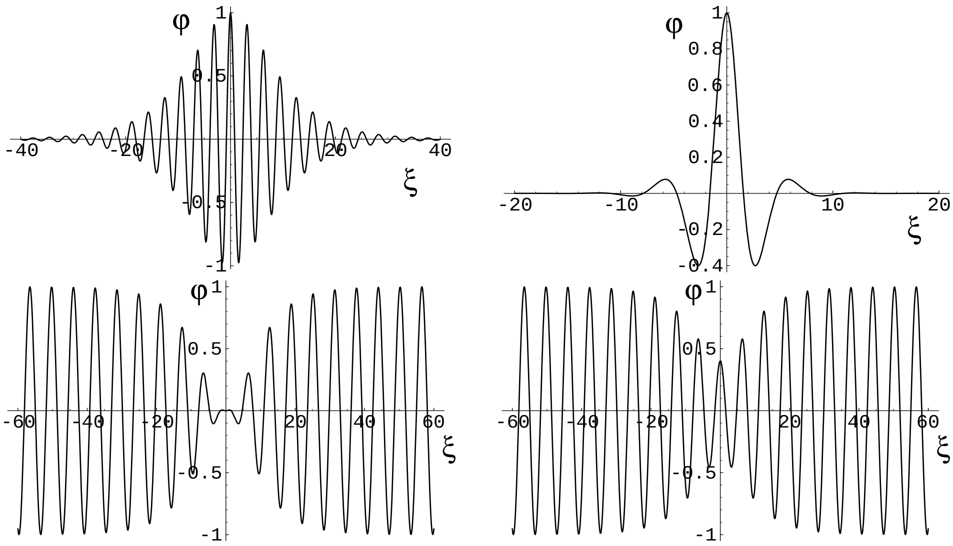

FIG. 1: A heuristic representation of wave packets modulated by solutions of the NLS equation. These envelope excitations are of the: (a, b) bright type ($PQ > 0$, pulses); (c) dark type, (d) gray type ($PQ < 0$, voids). Notice that the amplitude never reaches zero in (d).

value $\hat{k}_{cr} = \sqrt{2Q/P}\,|\hat{\psi}_{1,0}|$, i.e. for wavelengths above a threshold $\lambda_{cr} = 2\pi/\hat{k}_{cr}$; defining the instability growth rate $\sigma = |Im\hat{\omega}(\hat{k})|$, we see that it reaches its maximum value for $\hat{k} = \hat{k}_{cr}/\sqrt{2}$, viz.

$$\sigma_{max} = |Im\hat{\omega}|_{\hat{k}=\hat{k}_{cr}/\sqrt{2}} = |Q|\,|\hat{\psi}_{1,0}|^2 \,.$$

We see that the instability condition depends only on the sign of the product PQ, which may be studied numerically, relying on the expressions derived above.

B. Finite amplitude nonlinear excitations

The NLSE (22) is long known to possess distinct types of localized constant profile (solitary wave) solutions, depending on the sign of the product PQ [5, 15, 16]. Remember that this equation here describes the evolution of the wave's envelope, so these solutions represent slowly varying localized envelope structures, confining the (fast) carrier wave. The analytic form of these excitation can be found in the literature (see e.g. in [5] for a brief review) and need not be derived here in detail. Let us however briefly summarize those results.

Following Ref. [16], we may seek a solution of Eq. (22) in the form $\psi(\zeta,\tau) = \rho(Z,T)\,e^{i\,\Theta(\zeta,\tau)} + \text{c.c.}$, where ρ, σ are real variables which are determined by substituting into the NLSE and separating real and imaginary parts. The different types of solution thus obtained are summarized in the following.

For $PQ > 0$ we find the *(bright) envelope soliton*

$$\rho = \pm\rho_0\, sech\left(\frac{Z - u_e\,\tau}{L}\right), \quad \Theta = \frac{1}{2P}\left[u_e Z - (\Omega + \frac{1}{2}u_e^2)T\right], \tag{24}$$

which represents a localized pulse travelling at the envelope speed u_e and oscillating at a frequency Ω (at rest). The pulse width L depends on the maximum amplitude square ρ_0 as $L = (2P/Q)^{1/2}/\rho_0$. Since the product PQ is always positive for long wavelengths, as we saw above, this type of excitation will be rather privileged in dusty plasmas. The bright-type envelope soliton is depicted in Fig. 1a, b.

For $PQ < 0$, we obtain the *dark* envelope soliton (*hole*) [16]

$$\begin{aligned} \rho &= \pm\rho_1\left[1 - sech^2\left(\frac{Z - u_e T}{L'}\right)\right]^{1/2} \pm \rho_1\,\tanh\left(\frac{Z - u_e\,T}{L'}\right), \\ \Theta &= \frac{1}{2P}\left[u_e Z - \left(\frac{1}{2}u_e^2 - 2PQ\rho_1\right)T\right], \end{aligned} \tag{25}$$

which represents a localized region of negative wave density (shock) travelling at a speed u_e; see Fig. 1c. Again, the pulse width depends on the maximum amplitude square ρ_1 via $L' = (2|P/Q|)^{1/2}/\rho_1$.

Finally, still for $PQ < 0$, one also obtains the *gray* envelope solitary wave [16]

$$\rho = \pm\rho_2 \left[1 - a^2\, sech^2\left(\frac{Z - u_e T}{L''}\right)\right]^{1/2}, \tag{26}$$

which also represents a localized region of negative wave density. Comparing to the dark soliton (25), we note that the maximum amplitude ρ_2 is now finite (non-zero) everywhere; see Fig. 1d. The the pulse width of this gray-type excitation $L'' = \sqrt{2|P/Q|}/(a\,\rho_2)$ now also depends on an independent parameter a which represents the modulation depth ($0 < a \leq 1$). The lengthy expressions which determine the phase shift Θ and the parameter a, which are omitted here for brevity, can be found in Refs. [5, 16]. For $a = 1$, one recovers the *dark* soliton presented above.

An important qualitative result to be retained is that the envelope soliton width L and maximum amplitude ρ satisfy $L\rho \sim \sqrt{P/Q}$ (see above), and thus depend on (the ratio of) the coefficients P and Q; for instance, regions with higher values of P (or lower values of Q) will support wider (spatially more extended) localized excitations, for a given value of the maximum amplitude. Contrary to the KdV soliton picture, the width of these excitations does not depend on their velocity. It does, however, depend on the plasma parameters, e.g. here μ.

The localized envelope excitations presented above represent the slowly varying envelope which confines the (fast) carrier space and time oscillations, viz. $\phi = \Psi(X, Z)\cos(k\xi - \omega\tau)$ for the electric potential ϕ (and analogous expressions for the density n_i etc.; cf. (21)). The qualitative characteristics (width, amplitude) of these excitations, may be investigated by a numerical study of the ratio $P/Q \equiv \eta(k; \mu)$: recall that its sign determines the type (bright or dark) of the excitation, while its (absolute) value determines its width for a given amplitude (and vice versa). In Fig. 2 we have depicted the behaviour of η as a function of the wavenumber k and the parameter μ: higher values of μ correspond to lower curves. Remember that, for any given wavenumber k, the dust concentration (expressed via the value of μ) determines the soliton width L (for a given amplitude ρ; see discussion above) since $L \sim \eta^{1/2}/\rho$. Therefore, we see that the addition of *negative* dust generally ($\mu < 1$) results to higher values of η (i.e. wider or higher solitons), while *positive* dust ($\mu > 1$) has the opposite effect: it reduces the value of η (leading to narrower or shorter solitons). In a rather general manner, bright type solitons (pulses) seem to be rather privileged, since the ratio η (or the product PQ) of the coefficients P and Q is positive in most of the k, μ plane of values. One exception seems to be very the region of *very* low values of μ (typically below 0.2), which develops a negative tail of η for small k ($< 0.3\,k_D$): thus, a very high (> 80 per cent) electron depletion results in pulse destabilization in favour of dark-type excitations (Fig. 1c, d). Strictly speaking, η also becomes negative for very high wave number values ($> 2.5\,k_D$); nevertheless, we neglect – for rigor – this region from the analysis, in this (long wavelength λ) fluid picture (for a weak dust presence, short λ DIAWs may be quite strongly damped; however, this result may still be interesting for a strong presence of dust, when Landau damping is not a significant issue [1]).

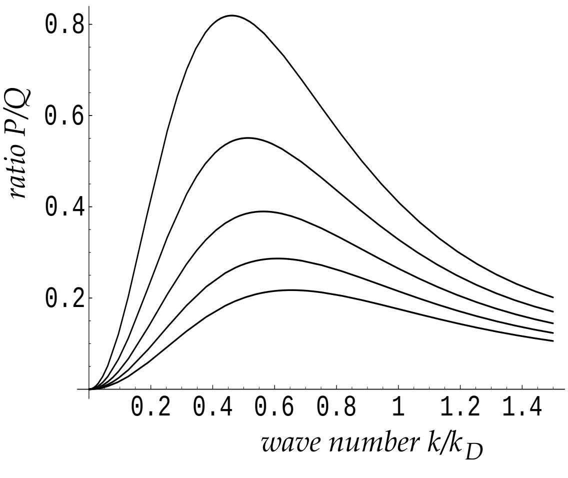

FIG. 2: The ratio $\eta = P/Q$ of the coefficients in the NLSE (22) is depicted versus the wave number k (normalized over k_D), for several values of the dust parameter μ; in descending order (from top to bottom): 0.8, 0.9, 1.0, 1.1, 1.2.

V. RELATION TO PREVIOUS WORKS: AN APPROXIMATE NONLINEAR LAGRANGEAN TREATMENT

By combining the Lagrangean system of Eqs. (12) to (16), one obtains the (reduced) evolution equation

$$\frac{\partial^2}{\partial \tau^2}\left(\frac{1}{n}\right) = -\frac{1}{n_0}\frac{\partial}{\partial \xi}\left[\frac{n}{n_0}\frac{\partial}{\partial \xi}\, g(w)\right], \tag{27}$$

where the function $g(x)$ [defined in Eq. (8)] is evaluated at

$$w(n) = n\left[1 - \delta^2 \frac{\partial^2}{\partial \tau^2}\left(\frac{1}{n}\right)\right].$$

Note that the ion density n has been scaled by its equilibrium value $n_{i,0}$, to be distinguished from the initial condition $n_0 = n(\xi, \tau = 0)$.

Despite its complex form, the nonlinear evolution equation (27) can be solved exactly by considering different special cases, as regards the order of magnitude of the dispersion–related parameter δ. This treatment, based on Ref. [9], will only be briefly summarized here, for the sake of reference.

First, one may consider very short scale variations, i.e. $L \ll \lambda_D$ (or $\delta \gg 1$). This amounts to neglecting collective effects, so oscillatory motion within a Debye sphere is essentially decoupled from neighboring ones. By considering $w(n) \approx -\delta^2\, n\, \partial^2(1/n)/\partial\tau^2$ and $\phi \approx 0$ (i.e. $\hat{n} \approx 1$), one may combine Eqs. (15) and (27) into

$$\left(\frac{\partial^2}{\partial \tau^2} + \omega_{p,i}^2\right)\left(\frac{1}{n} - 1\right) = 0\,, \tag{28}$$

which, imposing the initial condition $n(\xi, 0) = n_0(\xi)$, yields the solution

$$n(\xi, \tau) = \frac{n_0(\xi)}{\frac{n_0(\xi)}{n_{i,0}} + \left(1 - \frac{n_0(\xi)}{n_{i,0}}\right)\cos\omega_{p,i}\tau}\,. \tag{29}$$

Note that if the system is initially at equilibrium, viz. $n_0(\xi) = n_{i,0}$, then it remains so at all times $\tau > 0$. Now, one may go back to Eq. (12) and solve for $\alpha(\xi, \tau)$, which in turn immediately provides the mean fluid velocity u

$$u(\xi, \tau) = \omega_{p,i}\sin\omega_{p,i}\tau \int_{\xi_0}^{\xi}\left(1 - \frac{n_0(\xi')}{n_{i,0}}\right)d\xi'$$

via (10), and then $E(\xi, \tau)$ and $\phi(\xi, \tau)$. Finally, the variable transformation (9) may now be inverted, immediately providing the Eulerian position x in terms of ξ and τ. We shall not go into further details regarding this procedure, which is essentially analogue (yet not identical) to Davidson's treatment of electron plasma oscillations.

Quite interestingly, upon neglecting the dispersive effects, i.e. setting $\delta = 0$, Eq. (27) may be solved by separation of variables, and thus shown to possess a nonlinear special solution in the form of a product, say $n(\xi, \tau) = n_1(\xi) n_2(\tau)$ [13]. This calculation was put forward in Ref. [9] (where the study of IAW – in a single electron temperature plasma – was argued to rely on an equation quasi-identical to Eq. (27)). However, the solution thus obtained relies on doubtful physical grounds, since the assumption $\delta \approx 0$, which amounts to remaining close to equilibrium – cf. the last of Eqs. (6), implies an infinite space scale L (recall the definition of δ), contrary to the very nature of the (localized) nonlinear excitation itself. Rather not surprisingly, this solution was shown in Ref. [9] to decay fast in time, in both Eulerian and Lagrangean coordinates. Therefore, we shall not pursue this analysis any further.

VI. DISCUSSION AND CONCLUSIONS

We have studied the nonlinear propagation of dust ion acoustic waves propagating in a dusty plasma. By employing a Lagrangean formalism, we have investigated the modulational stability of the amplitude of the propagating dust ion acoustic oscillations and have shown that these electrostatic waves may become unstable, due to self interaction of the carrier wave. This instability may either lead to wave collapse or to wave energy localization, in the form of propagating localized envelope structures. We have provided an exact set of analytical expressions for these localized excitations.

This study complements similar investigations which relied on an Eulerian formulation of the dusty plasma fluid model [5]. In fact, the Lagrangean picture provides a strongly modified nonlinear stability profile for the wave amplitude, with respect to the previous (Eulerian) description; this was intuitively expected, since the passing to Lagrangean variables involves an inherently nonlinear transformation, which inevitably modifies the nonlinear evolution profile of the system described. However, the general qualitative result remains in tact: the dust ion acoustic-type electrostatic plasma waves may propagate in the form of localized envelope excitations, which are formed as a result of the mutual balance between dispersion and nonlinearity in the plasma fluid. More sophisticated descriptions, incorporating e.g. thermal or collisional effects, may be elaborated in order to refine the parameter range of the problem, and may be reported later.

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[1] P. K. Shukla and A. A. Mamun, *Introduction to Dusty Plasma Physics* (Institute of Physics Publishing Ltd., Bristol, 2002).
[2] F. Verheest, *Waves in Dusty Space Plasmas* (Kluwer Academic Publishers, Dordrecht, 2001).
[3] N. A. Krall and A. W. Trivelpiece, *Principles of plasma physics*, McGraw - Hill (New York, 1973); Th. Stix, *Waves in Plasmas*, American Institute of Physics (New York, 1992).
[4] For a review, see: P. K. Shukla and A. A. Mamun, New J. Phys. **5**, 17.1 (2003).
[5] I.Kourakis and P. K. Shukla, *Physics of Plasmas* **10** (9), 3459 (2003); *Eur. Phys. J. D* **28**, 109 (2003).
[6] R. C. Davidson and P. P. J. M. Schram, *Nuclear Fusion* **8**, 183 (1968).
[7] R. C. Davidson, *Methods in nonlinear plasma theory*, Academic Press (New York, 1972).
[8] E. Infeld and G. Rowlands, Phys. Rev. Lett. **58** (1987).
[9] N. Chakrabarti and M. S. Janaki, *Phys. Lett.* A **305** 393 (2002).
[10] N. Chakrabarti and M. S. Janaki, *Phys. Plasmas* **10** 3043 (2003).
[11] R. Z. Sagdeev, in *Reviews of Plasma Physics*, Vol. 4., ed. M. A. Leontovich, Consultants Bureau (New York, 1966), p.52.
[12] Eq. (12) was obtained from the (Lagrangean) density equation, which is reduced to: $\partial(n\alpha)/\partial\tau = 0$ by using the property (11); Eq. (12) then follows.
[13] Eqs. (19) are derived from Eqs. (12, 13, 14), $E = -\nabla\phi$ and (11), respectively. We have avoided the appearance of α^{-1} – cf. Eqs. (15, 16) – for analytical convenience.
[14] T. Taniuti and N. Yajima, J. Math. Phys. **10**, 1369 (1969); N. Asano, T. Taniuti and N. Yajima, J. Math. Phys. **10**, 2020 (1969).
[15] A. Hasegawa, *Plasma Instabilities and Nonlinear Effects* (Springer-Verlag, Berlin, 1975).
[16] R. Fedele, H. Schamel and P. K. Shukla, *Phys. Scripta* T **98** 18 (2002); R. Fedele and H. Schamel, *Eur. Phys. J. B* **27** 313 (2002); Fedele, *Phys. Scripta* **65** 502 (2002).