Nonlinear Modulated Envelope Electrostatic Wavepacket Propagation in Space and Laboratory Plasmas

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Abstract. A brief review of the occurrence of amplitude modulated structures in space and laboratory plasmas is provided, followed by a theoretical analysis of the mechanism of carrier wave (self-) interaction, with respect to electrostatic plasma modes. A generic collisionless unmagnetized fluid model is employed. Both cold– (zero-temperature) and warm– (finite temperature) fluid descriptions are considered and compared. The weakly nonlinear oscillation regime is investigated by applying a multiple scale (reductive perturbation) technique and a Nonlinear Schrödinger Equation (NLSE) is obtained, describing the evolution of the slowly varying wave amplitude in time and space. The amplitude's stability profile reveals the possibility of *modulational instability* to occur under the influence of external perturbations. The NLSE admits exact localized envelope (solitary wave) solutions of bright (*pulses*) or dark (*holes, voids*) type, whose characteristics depend on intrinsic plasma parameters. The role of perturbation obliqueness (with respect to the propagation direction), finite temperature and - possibly - defect (dust) concentration is explicitly considered. The relevance of this description with respect to known electron-ion (*e-i*) as well as *dusty* (complex) plasma modes is briefly discussed.

1. INTRODUCTION

Wave *amplitude modulation* (AM) is a generic nonlinear mechanism, known to dominate (finite amplitude) wave propagation in dispersive media. In a generic manner, the occurrence of AM, which is manifested as a slow variation of the wave's amplitude in space and time, may be due to parametric wave coupling, interaction between high- and low- frequency modes or self-interaction of the carrier wave (*auto-* or *self*-modulation). The relevance of this phenomenon with effects like secondary *harmonic generation* and *modulational instability* (MI), possibly leading to *energy localization* via localized *pulse formation*, is now long established in fields as diverse as Condensed Matter Physics, Nonlinear Optics and Biophysics [1 - 4]. As far as plasma modes are concerned [5, 6], the occurrence of such phenomena has been confirmed by experiments related to the nonlinear propagation of electrostatic (ES, e.g. ion-acoustic) [7 - 10] as well as electromagnetic (EM, e.g. whistler) waves; see Fig. 1. Early numerical simulations of electron cyclotron waves also predict such a behaviour [11]. In the context of Space Physics,

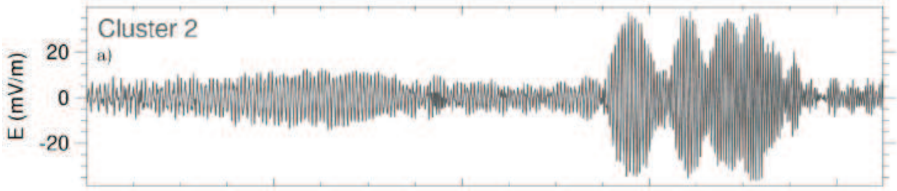

FIGURE 1. Modulated structures, related to *'chorus'* (EM) emission in the magnetosphere (CLUSTER satellite data; reprinted from [14]).

localized modulated wave packets are encountered in abundance e.g. in the Earth's magnetosphere, where they are associated with localized field and/or density variations observed during recent satellite missions [12 - 14] see e.g. Figs. 1 - 3. The occurrence of such wave forms is, for instance, thought to be related to the broadband electrostatic noise (BEN) encountered in the auroral region [12]. Furthermore, recent studies have supplied evidence for the relevance of such effects in dust-contaminated plasmas (*Dusty* or *Complex* Plasmas), where a strong presence of mesoscopic, massive, charged dust grains strongly affects the nonlinear and dispersive characteristics plasma as a medium [15, 16]. The modification of the plasma response due to the presence of the dust gives rise to new ES/EM modes, whose self-modulation was recently shown to lead to MI and soliton formation; these include e.g. the dust-acoustic (DA) [17 - 20] and dust-ion acoustic (DIAW) ES modes [18, 21 - 23] in addition to magnetized plasma modes, e.g. the Rao EM dust mode [24].

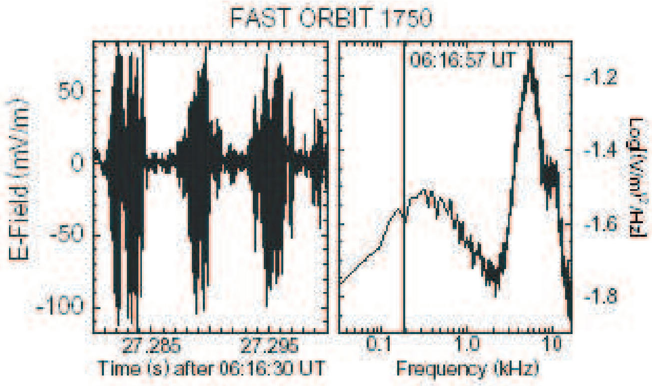

FIGURE 2. Electrostatic noise wave forms, related to modulated electron-acoustic waves (FAST satellite data; figure reprinted from [12]). The co-existence of a high (carrier) and a low (modulated envelope) frequencies is clearly reflected in the Fourier spectrum, in the right.

The purpose of this paper is to provide a generic methodological framework for the study of the nonlinear (self-)modulation of the amplitude of electrostatic (ES) plasma modes. The results provided in the following cover a variety of ES modes. The generic character of the nonlinear behaviour of these modes is meant to be emphasized, so focusing upon a specific mode is avoided on purpose. Where appropriate, details may

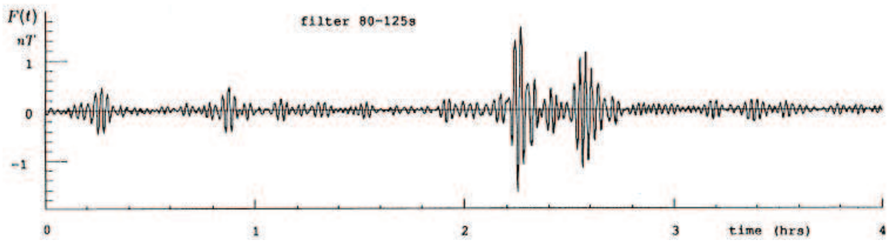

FIGURE 3. Localized envelope structures in the magnetosphere (reprinted from [13]).

be sought in the references [19, 22, 23, 25, 26], where some of this material was first presented.

2. THE MODEL: FORMULATION AND ANALYSIS

In a general manner, several known electrostatic plasma modes [5, 6] are modeled as propagating oscillations related to one dynamical plasma constituent, say α (mass m_α, charge $q_\alpha \equiv s_\alpha Z_\alpha e$; e is the absolute electron charge; $s = s_\alpha = q_\alpha/|q_\alpha| = \pm 1$ is the charge *sign*), against a background of one (or more) constituent(s), say α' (mass $m_{\alpha'}$, charge $q_{\alpha'} \equiv s_{\alpha'} Z_{\alpha'} e$, similarly). The background species is (are) often assumed to obey a known distribution, e.g. to be in a fixed (uniform) or in a thermalized (Maxwellian) state, for simplicity, depending on the particular aspects (e.g. frequency scales) of the physical system considered. For instance, the *ion-acoustic* (IA) mode refers to ions ($\alpha = i$) oscillating against a Maxwellian electron background ($\alpha' = e$) [5, 25], the *electron-acoustic* (EA) mode [5, 26] refers to electron oscillations ($\alpha = e$) against a fixed ion background ($\alpha' = i$), and so forth [5, 6]. As regards *dusty plasma* modes, the DA mode describes oscillations of dust grains ($\alpha = d$) against a Maxwellian electron and ion background ($\alpha' = e$, i) [16 - 19], while DIA waves denote IA oscillations in the presence of inertial dust in the background ($\alpha = i$, $\alpha' = e$, d) [16, 21 - 23]. Finally, this formalism readily applies in the case when a co-existence of two different populations of the same particle species occurs in the background, e.g. when two different electron temperatures are present ($\alpha' = c, h$, for *c*old and *h*ot electrons), affecting IA oscillations ($\alpha = i$) [25]; this situation is witnessed in the upper parts of the Earth's atmosphere.

2.1. A generic fluid description

A standard (single) fluid model for the dynamic species α consists of the moment evolution equations:

$$\begin{aligned} \frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \mathbf{u}_\alpha) &= 0 \\ \frac{\partial \mathbf{u}_\alpha}{\partial t} + \mathbf{u}_\alpha \cdot \nabla \mathbf{u}_\alpha &= -\frac{q_\alpha}{m_\alpha} \nabla \Phi - \frac{1}{m_\alpha n_\alpha} \nabla p_\alpha \end{aligned}$$

$$\frac{\partial p_\alpha}{\partial t} + \mathbf{u}_\alpha \cdot \nabla p_\alpha \;=\; -\gamma p_\alpha \nabla \cdot \mathbf{u}_\alpha \,, \tag{1}$$

where n_α, $\mathbf{u}_\alpha$ and p_α respectively denote the *density*, *mean* (fluid) *velocity* and *pressure* of species α. The electric potential Φ obeys Poisson's eq.:

$$\nabla^2\Phi \,=\, -4\pi \sum_{\alpha''=\alpha,\{\alpha'\}} q_{\alpha''}\, n_{\alpha''} \,=\, 4\pi e\,(n_e - Z_i n_i + ...) \tag{2}$$

Overall charge neutrality is assumed at equilibrium, i.e. $q_\alpha n_{\alpha,0} = -\sum_{\alpha'} q_{\alpha'} n_{\alpha',0}$. The parameter $\gamma = c_P/c_V = 1 + 2/f$ denotes the specific heat ratio (for f degrees of freedom).

By choosing appropriate scales for all quantities, the above system may be reduced to the following form:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) \;=\; 0\,, \tag{3}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \;=\; -s\nabla\phi - \frac{\sigma}{n}\nabla p\,, \tag{4}$$

$$\frac{\partial p}{\partial t} + \mathbf{u} \cdot \nabla p \;=\; -\gamma p \nabla \cdot \mathbf{u} \tag{5}$$

(the index α will be understood where omitted, viz. $s = s_\alpha$). The re-scaled (dimensionless) dynamic variables are now: $n = n_\alpha/n_{\alpha,0}$, $\mathbf{u} = \mathbf{u}_\alpha/c_*$, $p = p_\alpha/n_{\alpha,0}k_B T_\alpha$, and $\phi = |q_\alpha|\Phi/(k_B T_*)$, where $n_{\alpha,0}$ is the equilibrium density and $c_* = (k_B T_*/m_\alpha)^{1/2}$ is a characteristic (e.g. sound) velocity. Time and space are scaled over appropriately chosen scales t_0 [e.g. $\omega_{p,\alpha}^{-1} = (4\pi n_{\alpha,0} q_\alpha^2/m_\alpha)^{-1/2}$] and $r_0 = c_* t_0$; T_α is the fluid temperature, and T_* is an effective temperature (related to the background considered), to be determined for each problem under consideration (k_B is Boltzmann's constant). The temperature ratio T_α/T_* is denoted by σ, in this *warm model* [27] (the so-called *cold model* is recovered for $\sigma = 0$; see that Eq. (5) then becomes obsolete). The Lorentz force term was omitted, since wave propagation along the external magnetic field is considered. The system is closed by Poisson's equation, which may now be expressed as[1]

$$\nabla^2\phi \;=\; -s\left[n + \sum_{\alpha'} n_{\alpha'}\, q_{\alpha'}/(n_{\alpha,0}\, q_\alpha)\right] \;\equiv\; -s\,(n - \hat{n})\,. \tag{6}$$

Note that the neutralizing background (reduced) density

$$\hat{n} = -\sum_{\alpha'} \frac{n_{\alpha'}\, q_{\alpha'}}{n_{\alpha,0}\, q_\alpha} = -\frac{1}{s_\alpha Z_\alpha\, n_{\alpha,0}} \sum_{\alpha'} s_{\alpha'} Z_{\alpha'} n_{\alpha'} \tag{7}$$

is *a priori* [2] a function[3] of the potential ϕ; furthermore, it depends on the physical parameters (e.g. background temperature, plasma density, defect concentration, ...) involved in

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[1] A factor $\omega_{p,\alpha}^2 t_0^2$ is omitted in the right-hand side of Eq. (6), since equal to 1 for $t_0 = \omega_{p,\alpha}^{-1}$.
[2] This is only not true when the background is assumed fixed, e.g. for EA waves (i.e. $s_\alpha = -s_{\alpha'} = -1$, $n_{\alpha'} = n_i = const.$), where $\hat{n} = Z_i n_i/n_{e,0} = const.$
[3] Note that $\hat{n} = 1$ for $\phi = 0$, due to the equilibrium neutrality condition.

a given problem. The calculation in the specific paradigm of IA waves is explicitly provided below, for clarity.

2.2. Weakly nonlinear oscillation regime

What follows is essentially an implementation of the long known *reductive perturbation* technique [28 - 31], which was first applied in the study of electron plasma [28] and electron-cyclotron [11] waves, more than three decades ago.

Equations (3) - (5) and (9) form a system of evolution equations for the state vector[4] $\mathbf{S} = \{n, \mathbf{u}, p, \phi\}$ which accepts a harmonic (electrostatic) wave solution in the form $\mathbf{S} = \mathbf{S_0} \exp[i(k\mathbf{r} - \omega t)] + \mathrm{c.c.}$ Once the amplitude of this wave becomes non-negligible, a nonlinear harmonic generation mechanism enters into play: this is the first signature of nonlinearity, which manifests its presence once a slight departure from the weak-amplitude (linear) domain occurs. In order to study the (amplitude) modulational stability profile of these electrostatic waves, we consider small deviations from the equilibrium state $\mathbf{S}^{(0)} = (1, \mathbf{0}, 1, 0)^T$, viz. $\mathbf{S} = \mathbf{S}^{(0)} + \varepsilon \mathbf{S}^{(1)} + \varepsilon^2 \mathbf{S}^{(2)} + \ldots$, where $\varepsilon \ll 1$ is a smallness parameter. We have assumed that[5] $S_j^{(n)} = \sum_{l=-\infty}^{\infty} S_{j,l}^{(n)}(X, T)\, e^{il(k\mathbf{r} - \omega t)}$ (for $j = 1, 2, \ldots, d+3$ [4]; the condition $S_{j,-l}^{(n)} = S_{j,l}^{(n)\,*}$ holds, for reality). The wave amplitude is thus allowed to depend on the stretched (*slow*) coordinates $X = \varepsilon(x - \lambda)\, t$ and $T = \varepsilon^2 t$; the real variable $\lambda = \partial\omega(k)/\partial k_x \equiv \tilde{v}_g$ denotes the wave's *group velocity* along the modulation direction[6] x. The amplitude modulation direction is assumed *oblique* with respect to the (arbitrary) propagation direction[7], expressed by the wave vector $\mathbf{k} = (k_x, k_y) = (k\cos\theta, k\sin\theta)$. Accordingly, we set: $\partial/\partial t \to \partial/\partial t - \varepsilon\tilde{v}_g \partial/\partial X + \varepsilon^2 \partial/\partial T$, $\partial/\partial x \to \partial/\partial x + \varepsilon \partial/\partial X$ (while $\partial/\partial y$ remains unchanged) and $\nabla^2 \to \nabla^2 + 2\varepsilon\, \partial^2/\partial x \partial X + \varepsilon^2\, \partial^2/\partial X^2$, so that

$$\begin{aligned}
\frac{\partial}{\partial t} A_l^{(n)}\, e^{il\theta_1} &= \left(-\, il\omega A_l^{(n)} - \varepsilon\,\lambda\, \frac{\partial A_l^{(n)}}{\partial \zeta} + \varepsilon^2\, \frac{\partial A_l^{(n)}}{\partial \tau} \right) e^{il\theta_1}\,, \\
\nabla A_l^{(n)}\, e^{il\theta_1} &= \left(il\mathbf{k} A_l^{(n)} + \varepsilon\,\hat{x}\, \frac{\partial A_l^{(n)}}{\partial \zeta} \right) e^{il\theta_1}\,, \\
\nabla^2 A_l^{(n)}\, e^{il\theta_1} &= \left(-\, l^2 k^2 A_l^{(n)} + 2\varepsilon\, ilk_x\, \frac{\partial A_l^{(n)}}{\partial \zeta} + \varepsilon^2\, \frac{\partial^2 A_l^{(n)}}{\partial \zeta^2} \right) e^{il\theta_1}
\end{aligned} \tag{8}$$

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[4] Note that $\mathbf{S} \in \Re^{d+3}$ in a $d-$ dimensional problem ($d = 1, 2, 3$).

[5] In practice, only terms with $l \leq n$ do contribute in this summation. This simply means that up to 1st harmonics are expected for $n = 1$, up to 2nd phase harmonics for $n = 2$, and so forth.

[6] This is a - physically expected - constraint which is imposed by the equations for $n = 2$ and $l = 1$ (1st harmonics at 2nd order). Alternatively, one may assume a dependence on $X_n = \varepsilon^n x$ (plus a similar expansion for y, z and t) for $n = 0, 1, 2, \ldots$; the condition for annihilation of secular terms then reads: $\partial A_1^{(1)}/\partial T_1 + (\partial\omega/\partial k_x)\partial A_1^{(1)}/\partial X_1$, i.e. $A_1^{(1)} = A_1^{(1)}(X_1 - \tilde{v}_g T_1)$ (for any of the 1st harmonic amplitudes $A_1^{(1)} \in \{S_{1,j}^{(1)}\}$), which essentially amounts to the same constraint.

[7] Cf. Refs. [32, 33], where a similar treatment is adopted.

for any $l-$th phase harmonic amplitude $A_l^{(n)}$ among the components of $\mathbf{S}_l^{(n)}$; obviously, θ_1 here denotes the elementary phase $\theta_1 \equiv \mathbf{kr} - \omega t$.

By expanding near $\phi \approx 0$, Poisson's eq. may formally be cast in the form

$$\nabla^2\phi = \phi - \alpha\,\phi^2 + \alpha'\,\phi^3 - s\,\beta\,(n-1)\,, \tag{9}$$

where the exact form of the real coefficients α, α' and β (to be distinguished from the species indices above, obviously) are to be determined exactly for any specific problem, and contain all the essential dependence on the plasma parameters. Note that the right-hand side in Eq. (9) cancels at equilibrium.

A case study: ion-acoustic waves. In order to make our method and notation clear, let us explicitly consider the simple case of *ions* ($\alpha = i$ and $q_\alpha = q_i = +Z_i e$, i.e. $s_\alpha = +1$) oscillating against thermalized *electrons* ($\alpha' = e$ and $q_{\alpha'} = q_e = -e$, i.e. $s_{\alpha'} = -1$; $n_e = n_{e,0} e^{e\Phi/(k_B T_e)}$). Adopting the scaling defined above, and using the equilibrium neutrality condition $n_{e,0} = Z_i n_{e,0}$, it is a trivial exercise to cast Poisson's Eq. (2) into the (reduced) form: $\nabla^2\phi = -(\omega_{p,i}\, r_0/c_*)^2\,[n - e^{T_*\phi/(Z_i T_e)}] \equiv -(n - e^{\xi\phi})$, where we took: $t_0 = r_0/c_* = \omega_{p,i}^{-1}$ and $\xi \equiv T_*/(Z_i T_e)$. Now, expanding near $\phi \approx 0$, viz. $e^{\xi\phi} \approx 1 + \xi\phi + \xi^2\phi^2/2 + \xi^3\phi^3/6 + ...$, we have: $\nabla^2\phi \approx \xi\phi + \xi^2\phi^2/2 + \xi^3\phi^3/6 - (n-1)$. Finally, setting the temperature scale T_* equal to $T_* = Z_i T_e$, for convenience (viz. $\xi = 1$)[8], one recovers exactly Eq. (9) with $\alpha = -1/2$, $\alpha' = 1/6$ and $\beta = 1$. It may be noted that this case has been studied, both for parallel modulation ($\theta = 0$), in Ref. [29], and for oblique modulation, in Refs. [32]; those results are recovered from the formulae below.

Amplitude evolution equations. By substituting into Eqs. (3) - (5) and (9) and isolating distinct orders in ε, we obtain the nth-order reduced equations

$$-il\omega n_l^{(n)} + il\mathbf{k}\cdot\mathbf{u}_l^{(n)} - \lambda\,\frac{\partial n_l^{(n-1)}}{\partial\zeta} + \frac{\partial n_l^{(n-2)}}{\partial\tau} + \frac{\partial u_{l,x}^{(n-1)}}{\partial\zeta}$$
$$+ \sum_{n'=1}^{\infty}\sum_{l'=-\infty}^{\infty}\left[il\mathbf{k}\cdot\mathbf{u}_{l-l'}^{(n-n')}\,n_{l'}^{(n')} + \frac{\partial}{\partial\zeta}\left(n_{l'}^{(n')}\,u_{(l-l'),x}^{(n-n'-1)}\right)\right] = 0\,, \tag{10}$$

$$-il\omega\mathbf{u}_l^{(n)} + s\,il\mathbf{k}\phi_l^{(n)} - \lambda\,\frac{\partial\mathbf{u}_l^{(n-1)}}{\partial\zeta} + \frac{\partial\mathbf{u}_l^{(n-2)}}{\partial\tau} + s\,\frac{\partial\phi_l^{(n-1)}}{\partial\zeta}\hat{x}$$
$$+ \sum_{n'=1}^{\infty}\sum_{l'=-\infty}^{\infty}\left[il'\mathbf{k}\cdot\mathbf{u}_{l-l'}^{(n-n')}\,\mathbf{u}_{l'}^{(n')} + u_{(l-l'),x}^{(n-n'-1)}\,\frac{\partial\mathbf{u}_{l'}^{(n')}}{\partial\zeta}\right] + \sigma\left(ilp_l^{(n)}\,\mathbf{k} + \frac{\partial p_l^{(n-1)}}{\partial\zeta}\,\hat{x}\right)$$
$$+ \sum_{n'=1}^{\infty}\sum_{l'=-\infty}^{\infty} n_{(l-l')}^{(n-n')}\left\{-il'\omega\mathbf{u}_{l'}^{(n')} + s\,il'\mathbf{k}\phi_{l'}^{(n')} - \lambda\,\frac{\partial\mathbf{u}_{l'}^{(n'-1)}}{\partial\zeta} + \frac{\partial\mathbf{u}_{l'}^{(n'-2)}}{\partial\tau}\right.$$

[8] Note that a different choice for T_* would lead to a modified right-hand-side in Eq. (9), i.e. a factor $\xi \neq 1$ would precede the first term (in ϕ). This might, of course, also be a legitimate choice of scaling; however, the following formula are not valid - and should be appropriately modified - in this case. Obviously though, the qualitative results of this study are not affected by the choice of scaling.

$$+s\frac{\partial\phi_{l'}^{(n'-1)}}{\partial\zeta}\hat{x}+\sum_{n''=1}^{\infty}\sum_{l''=-\infty}^{\infty}\left[il''\mathbf{k}\cdot\mathbf{u}_{l'-l''}^{(n'-n'')}\mathbf{u}_{l''}^{(n'')}+u_{(l'-l''),x}^{(n'-n''-1)}\frac{\partial\mathbf{u}_{l''}^{(n'')}}{\partial\zeta}\right]\Bigg\}=0, \quad (11)$$

$$-il\omega p_l^{(n)}+il\gamma\mathbf{k}\cdot\mathbf{u}_l^{(n)}-\lambda\frac{\partial p_l^{(n-1)}}{\partial\zeta}+\frac{\partial p_l^{(n-2)}}{\partial\tau}+\gamma\frac{\partial u_{l,x}^{(n-1)}}{\partial\zeta}$$
$$+\gamma\sum_{n'=1}^{\infty}\sum_{l'=-\infty}^{\infty}p_{l-l'}^{(n-n')}\left(il'\mathbf{k}\cdot\mathbf{u}_{l'}^{(n')}+\frac{\partial u_{l',x}^{(n'-1)}}{\partial\zeta}\right)$$
$$+\sum_{n'=1}^{\infty}\sum_{l'=-\infty}^{\infty}\left(il'\mathbf{k}\cdot\mathbf{u}_{l-l'}^{(n-n')}p_{l'}^{(n')}+\frac{\partial u_{l'}^{(n'-1)}}{\partial\zeta}u_{(l-l'),x}^{(n-n')}\right)=0,$$

$$-(l^2k^2+1)\,\phi_l^{(n)}+s\beta\,n_l^{(n)}+2ilk_x\frac{\partial\phi_l^{(n-1)}}{\partial\zeta}+\frac{\partial^2\phi_l^{(n-2)}}{\partial\zeta^2}$$
$$+\alpha\sum_{n'=1}^{\infty}\sum_{l'=-\infty}^{\infty}\phi_{l-l'}^{(n-n')}\phi_{l'}^{(n')}-\alpha'\sum_{n',n''=1}^{\infty}\sum_{l',l''=-\infty}^{\infty}\phi_{l-l'-l''}^{(n-n'-n'')}\phi_{l'}^{(n')}\phi_{l''}^{(n'')}=0. \quad (12)$$

Upon substituting into Eqs. (3) - (5) and (9), one is left with the task of isolating orders in ε^n (i.e. $n=1,2,...$) and successively solving for the harmonic amplitudes $S_{j,l}^{(n)}$. The calculation, particularly lengthy yet perfectly straightforward, can be found e.g. in [22] for IA ($s=+1$) and in [26] for EA waves ($s=-1$).

The first harmonic amplitudes are determined (to order $\sim\varepsilon^1$) as

$$n_1^{(1)}=s[(1+k^2)/\beta]\,\psi=[k/(\omega\cos\theta)]\,u_{1,x}^{(1)}=[k/(\omega\sin\theta)]\,u_{1,y}^{(1)}=p_1^{(1)}/\gamma \quad (13)$$

in terms e.g. of the potential correction $\phi_1^{(1)}\equiv\psi$, along with the dispersion relation $\omega^2=\beta k^2/(k^2+1)+\gamma\sigma k^2$. Furthermore, the amplitudes of the 2nd and 0th (constant) harmonic corrections are obtained in $\sim\varepsilon^2$; the lengthy expressions are omitted here, for brevity[9].

The envelope evolution equation. The potential correction ψ is found to obey a compatibility condition in the form of a *nonlinear Schrödinger–type equation* (NLSE)

$$i\frac{\partial\psi}{\partial T}+P\frac{\partial^2\psi}{\partial X^2}+Q\,|\psi|^2\,\psi=0. \quad (14)$$

Both the *dispersion coefficient* P, in fact related to the curvature of the dispersion curve as $P=\partial^2\omega/2\partial k_x^2=\left[\omega''(k)\cos^2\theta+\omega'(k)\sin^2\theta/k\right]/2$, and the *nonlinearity coefficient* Q, which is due to carrier wave self-interaction, are functions of k, θ and β, as expected (in addition to α,α', for Q). The exact general expressions obtained (reported in the Appendix) may be *tailor fit* to any given electrostatic plasma wave problem (via the

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[9] The exact expressions for the 2nd order solution can be found e.g. in Ref. [19]; please refer to Eqs. (21) - (26) therein, which are exactly valid here, as they stand.

form of the parameters α, α', β), in view of a numerical investigation of the wave's amplitude dynamics (e.g. stability profile, wave localization; see in the following).

3. AMPLITUDE STABILITY PROFILE

It is known (see e.g. in Refs. [1 - 4, 34] that the evolution of a wave whose amplitude obeys Eq. (14) depends on the coefficient product PQ, which may be investigated in terms of the physical parameters involved. To see this, first check that Eq. (14) supports the plane (Stokes') wave solution $\psi = \psi_0 \exp(iQ|\psi_0|^2 T)$; the standard linear analysis consists in perturbing the amplitude by setting: $\hat{\psi} = \hat{\psi}_0 + \varepsilon\, \hat{\psi}_{1,0} \cos(\tilde{k}X - \tilde{\omega}T)$. One thus obtains the (perturbation) dispersion relation: $\tilde{\omega}^2 = P\tilde{k}^2 \left(P\tilde{k}^2 - 2Q|\hat{\psi}_{1,0}|^2\right)$. One immediately sees that if $PQ > 0$, the amplitude ψ is *unstable* for $\tilde{k} < \sqrt{2Q/P}|\psi_{1,0}|$. If $PQ < 0$, the amplitude ψ is *stable* to external perturbations.

This formalism allows for a numerical investigation of the stability profile in terms of parameters e.g. like wavenumber k, (oblique) perturbation angle α, temperature T_α, background plasma parameters etc. In figure 4, we have depicted the region $PQ < 0$ ($PQ > 0$) in black (white) color, for IA waves; see in Ref. [23] for details.

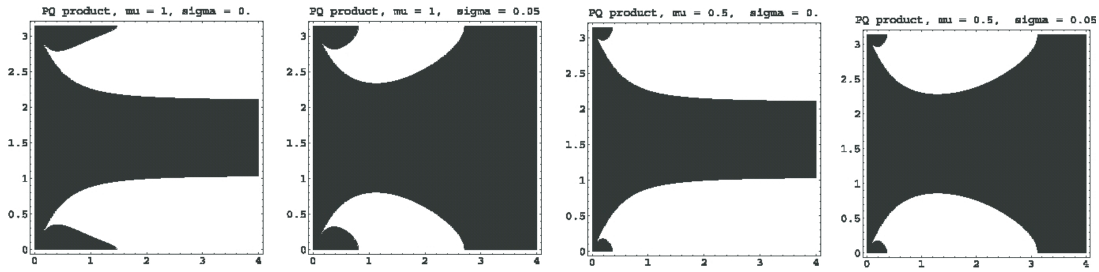

FIGURE 4. The region of positive (negative) values of the product PQ are depicted in white (black), in the *wavenumber k - modulation angle* α plane. The first two plots refer to IA waves: $\sigma = 0$ (cold model); $\sigma = 0.05$ (warm model). Similar for the latter two, but for DIA waves (see in the text); we have taken a *negative* dust density: $\mu = n_{e,0}/(Z_i n_{i,0}) = 0.5$ (from Ref. [23]). The dust presence strongly modifies the stability profile (rather enhancing instability here).

4. ENVELOPE EXCITATIONS

The modulated (electrostatic potential) wave resulting from the above analysis is of the form $\phi_1^{(1)} = \varepsilon\hat{\psi}_0 \cos(\mathbf{kr} - \omega t + \Theta) + \mathscr{O}(\varepsilon^2)$, where the slowly varying amplitude[10] $\psi_0(X,T)$ and phase correction $\Theta(X,T)$ (both real functions of $\{X,T\}$; see [35] for details) are determined by (solving) Eq. (14) for $\psi = \psi_0 \exp(i\Theta)$. The different types of solution thus obtained are summarized in the following.

[10] In fact, the potential correction amplitude here is $\hat{\psi}_0 = 2\psi_0$, from Euler's formula: $e^{ix} + e^{-ix} = 2\cos x$ ($x \in \Re$). Note that once the potential correction $\phi_1^{(1)}$ is determined, density, velocity and pressure corrections follow from (13).

Bright-type envelope solitons. For *positive PQ*, the carrier wave is modulationally *unstable*; it may either *collapse*, due to external perturbations, or lead to the formation of *bright* envelope modulated wavepackets, i.e. localized envelope *pulses* confining the carrier (see Fig. 5)[11]:

$$\psi_0 = \left(\frac{2P}{QL^2}\right)^{1/2} \operatorname{sech}\left(\frac{X - v_e T}{L}\right), \quad \Theta = \frac{1}{2P}\left[v_e X + \left(\Omega - \frac{v_e^2}{2}\right)T\right] \tag{15}$$

[35], where v_e is the envelope velocity; L and Ω represent the pulse's spatial width and oscillation frequency (at rest), respectively. We note that L and ψ_0 satisfy $L\psi_0 = (2P/Q)^{1/2}$ = constant [in contrast with Korteweg-deVries (KdV) solitons, where $L^2\psi_0$ = const. instead]. Also, the amplitude ψ_0 is independent from the velocity v_e here.

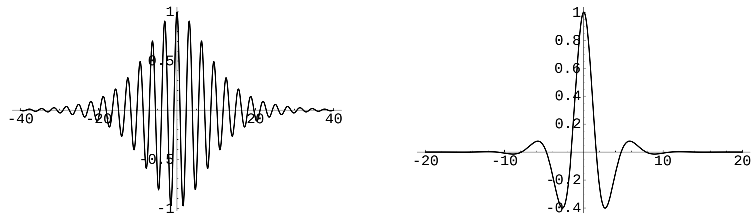

FIGURE 5. *Bright* type modulated wavepackets (for $PQ > 0$), for two different (arbitrary) sets of parameter values.

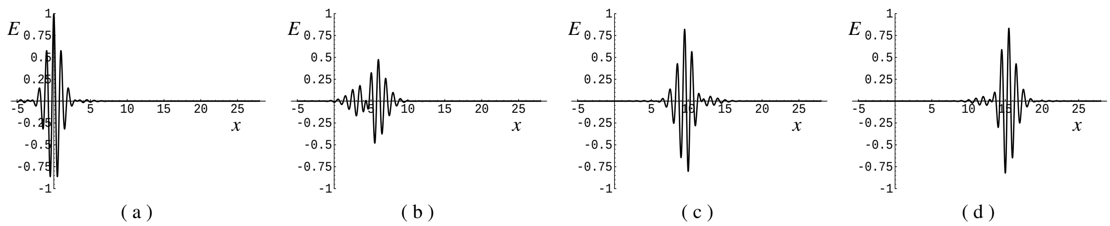

FIGURE 6. Bright envelope soliton propagation, at different times $t_1 < \cdots < t_4$ (arbitrary parameter values): cf. the structures encountered in satellite observations, e.g. see Fig. 3.

Dark-type envelope solitons. For $PQ < 0$, the carrier wave is modulationally *stable* and may propagate as a *dark/grey* envelope wavepackets, i.e. a propagating localized *hole* (a *void*) amidst a uniform wave energy region. The exact expression for *dark* envelopes reads[11] [35]:

$$\psi_0 = \psi'_0 \left|\tanh\left(\frac{X - v_e T}{L'}\right)\right|, \qquad \Theta = \frac{1}{2P}\left[v_e X + \left(2PQ{\psi'_0}^2 - \frac{v_e^2}{2}\right)T\right] \tag{16}$$

(see Fig. 7a); again, $L'\psi'_0 = (2|P/Q|)^{1/2}$ (=cst.).

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[11] These expressions are readily obtained from Ref. [35], by shifting the variables therein to our notation as: $x \to \zeta,\ s \to \tau,\ \rho_m \to \rho_0,\ \alpha \to 2P,\ q_0 \to -2PQ,\ \Delta \to L,\ E \to \Omega,\ V_0 \to u$.

Grey-type envelope solitons. The *grey*-type envelope (also obtained for $PQ < 0$) reads[11] [35]:

$$\psi_0 = \psi''_0 \{1 - d^2 \operatorname{sech}^2\{[X - v_e T]/L''\}\}^{1/2},$$

$$\Theta = \frac{1}{2P}\left[V_0 X - \left(\frac{1}{2}V_0^2 - 2PQ\psi''^2_0\right)T + \Theta_0\right] - S\sin^{-1}\frac{d\tanh\left(\frac{X-v_e T}{L''}\right)}{\left[1 - d^2\operatorname{sech}^2\left(\frac{X-v_e T}{L''}\right)\right]^{1/2}}. \quad (17)$$

Here Θ_0 is a constant phase; S denotes the product $S = \operatorname{sign}(P) \times \operatorname{sign}(v_e - V_0)$. The pulse width $L'' = (|P/Q|)^{1/2}/(d\psi''_0)$ now also depends on the real parameter d, given by: $d^2 = 1 + (v_e - V_0)^2/(2PQ\psi''^2_0) \leq 1$. $V_0 = \text{const.} \in \Re$ satisfies[11]: $V_0 - \sqrt{2|PQ|\,\psi''^2_0} \leq v_e \leq V_0 + \sqrt{2|PQ|\,\psi''^2_0}$. For $d = 1$ (thus $V_0 = v_e$), one recovers the *dark* envelope soliton.

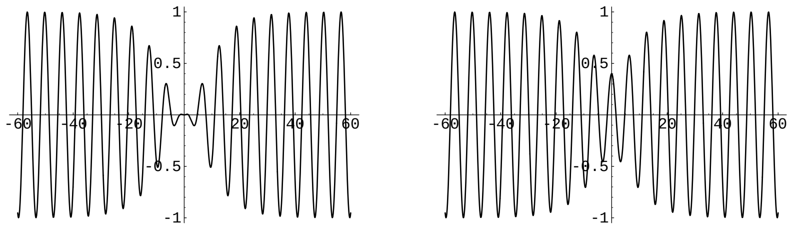

FIGURE 7. *Dark* (left) and *grey* (right) type modulated wavepacket (for $PQ < 0$). See that the amplitude never reaches zero in the latter case.

5. CONCLUSION

This work has been devoted to the study of the conditions for occurrence of *modulational instability*, related to the formation of *envelope localized structures*, with respect to electrostatic waves propagating in an unmagnetized plasma. We have shown that the envelope modulated electrostatic wave packets which are widely observed during satellite missions and laboratory experiments, may be efficiently modeled by making use of a *reductive perturbation* (multiple scales) technique [28]. Explicit criteria are thus obtained, which determine the wave's modulational stability profile and predict the occurrence of localized envelope excitations of either bright or dark/grey type. This methodology allows for an investigation of the nonlinear modulational profile of a (any) given electrostatic mode. Generalization in the presence of a magnetic field is on the way and will be reported soon.

ACKNOWLEDGMENTS

Support from the *Deutsche Forschunsgemeinschaft* through *Sonderforschungsbereich 591 – Universelles Verhalten gleichgewichtsferner Plasmen: Heizung, Transport und*

Strukturbildung, as well as from the EU (Contract No. HPRN-CT-2000-00140), is acknowledged. Figures 1, 2, 3 and 4 are reprinted from Refs. [14, 12, 13, 23] respectively.

I.K. expresses his gratitude to the organizers of the *22nd Summer School and Intern. Symposium on the Physics of Ionized Gases* (SPIG 2004; 23 - 27 August 2004, Tara Park, Serbia and Montenegro), and to Prof. Dusan Jovanoviç in particular, for an invitation to present this work and for the unprecedented hospitality provided during his stay.

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A. (APPENDIX) THE COEFFICIENTS P AND Q IN EQ. (14)

The dispersion coefficient P in Eq. (14) is equal to $P = (\partial^2\omega(k)/\partial k_x^2)/2$; it is given by:

$$P(k) = \frac{1}{\beta}\frac{1}{2\omega}\left(\frac{\omega}{k}\right)^4 \left[v_1 - \left(v_1 + 3\frac{v_2}{\beta}\,\omega^2\right)\cos^2\theta\right], \qquad (18)$$

where we have defined:

$$v_1 = \beta\,\frac{\beta+\sigma\gamma(1+k^2)^2}{[\beta+\sigma\gamma(1+k^2)]^2}, \qquad v_2 = \beta^3\,\frac{3\beta+\gamma\sigma(3-k^2)(1+k^2)}{3\,[\beta+\gamma\sigma\,(1+k^2)]^4} \qquad (19)$$

(see that $v_{1,2} \to 1$ when $\sigma \to 0$).

The nonlinearity coefficient Q in Eq. (14) is due to the carrier wave self-interaction; it is given by: $Q = \sum_{j=0}^{4} Q_j$, where:

$$\begin{aligned}
Q_0 \;=\;& +\frac{1}{2\omega}\frac{1}{\beta^2}\frac{1}{(1+k^2)^2}\frac{1}{\beta+\gamma\sigma-v_g^2}\times \\
&\Big\{\beta\,k^2\Big[\beta\left[3+6k^2+4k^4+k^6+2\,\alpha\,\beta\left(s\,(2k^2+3)+2\,\alpha\,v_g^2\right)\right] \\
&+\gamma\sigma\left[(\gamma+1)\,(1+k^2)^3+2\,\alpha\,\beta\left(-2\alpha\beta+s\,\gamma(1+k^2)^2\right)\right] \\
&+\left[\beta\,(2+4k^2+3k^4+k^6+2s\alpha\beta)+2\gamma\sigma\,(1+k^2)^2\,(1+k^2+s\alpha\beta)\right]\cos 2\theta\Big] \\
&+2\,(1+k^2)^4\,(\beta+\gamma\sigma)\,\omega^2\cos^2\theta + k\,(1+k^2)\left[\beta k^2+\omega^2\,(1+k^2)\right]\frac{v_g}{\omega}\times \\
&\quad\left[\beta\,(1+k^2+2s\alpha\beta)+\gamma(\gamma-1)\,\sigma\,(1+k^2)^2\right]\cos\theta\Big\}, \qquad (20)\\
Q_1 \;=\;& \frac{3\,\alpha'\beta}{2\,\omega}\frac{k^2}{(1+k^2)^2}, \qquad (21)\\
Q_2 \;=\;& -\frac{1}{12\beta^3}\frac{1}{\omega}\frac{1}{k^2\,(1+k^2)^2}\Big\{2\beta\,k^2\Big[5\,s\,\alpha\,\beta^2\,(1+k^2)^2+2\alpha^2\beta^3 \\
&+2\gamma^2\,\sigma\,(1+k^2)^4\,(1+4k^2)+\beta\,(1+k^2)^3\,(3+9k^2+2\,s\,\alpha\,\gamma^2\,\sigma)\Big] \\
&+(1+k^2)^3\omega^2\Big[\beta\,(3+9k^2+6k^4+2s\alpha\beta)+2\gamma^2\,\sigma\,(1+k^2)^2\,(1+4k^2)\Big]\Big\}. \qquad (22)
\end{aligned}$$

The coefficients Q_3 and Q_4 are related to finite temperature effects (in the warm model) and cancel for $\sigma = 0$. They can be directly computed from (36) - (39) in Ref. [19][12]; the lengthy final expressions are omitted here, for brevity.

[12] All the expressions from (9) and on in in Ref. [19] are exactly valid here; this is due to the generic form of the reduced Eqs. (3) - (5) and (9) of the model.