1. Introduction

Modulation instability (MI) is a well-known mechanism of energy localization dominating propagation in nonlinear dispersive media, has been widely investigated in the past, with respect to plasma electrostatic modes, e.g., ion-acoustic waves (IAW), and experiments have confirmed these studies [1].

The purpose of this study is to provide a generic methodological framework for the study of the nonlinear (self)-modulation of the amplitude of such electrostatic modes, a mechanism known to be associated with harmonic generation and the formation of localized envelope modulated wave packets, such as the ones abundantly observed during laboratory experiments and satellite observations, e.g., in the Earth’s magnetosphere.

2. The model: a generic description

In general, several known electrostatic plasma modes [2] consist of propagating oscillations of one dynamical plasma constituent, say $\alpha$ (mass $m_\alpha$, charge $q_\alpha$, etc.); the absolute electron charge, $e = q_\alpha = q_\beta = \pm 1$ is the charge sign, $\beta$ against a background of one (or more) constituent(s); $\alpha'$ (mass $m_{\alpha'}$, charge $q_{\alpha'}$, etc.) is the latter (i.e.) often assumed to obey a known distribution, e.g., being in a fixed (uniform) $n_{\alpha'} = const$ or in a thermalized (Maxwellian) state $n_{\alpha'} \propto e^{-\beta \alpha' T}$ ($T$ is the temperature of species $\alpha'$, e, i, ...), for simplicity, depending on the particular aspects (e.g., frequency spectrum) of the physical system considered. For instance,

- the ion-acoustic (IA) mode refers to ions ($\alpha = i$) oscillating against a Maxwellian electron background of $\alpha' = e$;

- the electron-acoustic (EA) mode refers to oscillations ($\alpha = e$) against a fixed ion background of $\alpha' = i$.

The standard (single) fluid model for the inertial species $\alpha$ provides the momentum evolution equations:

$$\frac{d}{dt} \rho_\alpha \mathbf{v}_\alpha = -\nabla \mathbf{\Phi}_\alpha - q_\alpha \mathbf{E} = 0,$$

$$\frac{d}{dt} \mathbf{E} = -\nabla \Phi = \nabla \int \frac{q_\alpha^2}{2} \mathbf{v}_\alpha^2 \mathbf{d} \mathbf{x},$$

also $\nabla \Phi = \alpha - \alpha'^2 + \alpha'^2 \mathbf{v}_\alpha^2 = (n - 1) \mathbf{v}_\alpha$.

i.e., Poison's Eq. $\nabla \Phi = -\nabla \int \frac{q_\alpha^2}{2} \mathbf{v}_\alpha^2 \mathbf{d} \mathbf{x}$.

Overall neutrality is assumed at equilibrium: $\sum q_\alpha n_\alpha = 0 = n_{\alpha'} - n_{\alpha''}. $

We have defined the reduced (dimensionless) quantities: $\nu = n_{\alpha''} / n_{\alpha}$; $\tau = \tau_{\alpha''} / \tau_{\alpha}$; $\nu_0 = \tau_0 / \tau$; $\nu = p_{\alpha''} \nu_0 / (p_{\alpha} \nu_0 + p_{\alpha''} \nu_0)$; $\nu = \nu_{\alpha''} \mathbf{v}_{\alpha''} \mathbf{v}_{\alpha''}$; $\nu = \nu_{\alpha'} \mathbf{v}_{\alpha'} \mathbf{v}_{\alpha'}$.

The electric potential $\mathbf{E} = \nabla \Phi \approx \nabla \int \frac{q_\alpha^2}{2} \mathbf{v}_\alpha^2 \mathbf{d} \mathbf{x}$.

$\nu = \nu_{\alpha''} \mathbf{v}_{\alpha''} \mathbf{v}_{\alpha''}$, $\nu = \nu_{\alpha'} \mathbf{v}_{\alpha'} \mathbf{v}_{\alpha'}$.

2. Multiple scales (reductive) perturbation method.

Let $S$ be the state (column) vector ($n, u, p, \alpha^T$); the equilibrium state is $S_0 = (1, 0, 1, 0)$. We shall consider small deviations by taking ($\epsilon < 1$)

$$S = S_0 + \epsilon S_1 + \epsilon^2 S_2 + \ldots + \epsilon^n S_n.$$  

$S_1$ is the structure (column) vector ($n, u, p, \alpha^T$); equilibrium state is $S_0 = (1, 0, 1, 0)$. We shall consider small deviations by taking ($\epsilon < 1$)

$$S = S_0 + \epsilon S_1 + \epsilon^2 S_2 + \ldots + \epsilon^n S_n.$$  

We define the stretched (slow) space and time variables $[\xi, \tilde{\tau}] = \epsilon \tau$; $\tau = \epsilon^{-1} (\xi, \tilde{\tau})$, the fast carrier phase is $\theta = \frac{\theta}{\epsilon}$; $\epsilon = \frac{\epsilon}{\epsilon}$; $\epsilon = \frac{\epsilon}{\epsilon}$.

The harmonic amplitudes vary slowly along $x$: $S_{\alpha'}(\xi, \tilde{\tau}) = S_{\alpha'}^0(\xi, \tilde{\tau})$, wave number $k = (k_x, k_y) = (k \cos \theta, k \sin \theta)$.


Substituting into (2), one obtains, successively (details in [2]):

- the first harmonic of the perturbation:

$$n_1 = 1 + k_x^2 + k_y^2, \quad \tau_1 = 1 + k_x^2 + k_y^2,$$

- the compatibility condition (dispersion relation):

$$\nu^2 = \frac{k_x^2}{k_x^2 + \nu_0} + \frac{k_y^2}{k_y^2 + \nu_0}.$$  

- the 2nd order contributions:

$S_{\alpha''} = S_{\alpha''}^0 + \epsilon S_{\alpha''}^1 + \epsilon^2 S_{\alpha''}^2 + \ldots$.

4. Derivation of the Nonlinear Schrödinger Equation

Proceeding to order $\epsilon^2$, the equations for $\epsilon^2$ is a yield an explicit compatibility condition i.e. the Nonlinear Schrödinger Equation

$$\frac{\partial}{\partial \tilde{\tau}} \psi + \frac{1}{2} \frac{\partial}{\partial \xi} \left( \frac{\partial^2 \psi}{\partial \xi^2} \right) - \frac{\nu}{\nu_0} \psi = 0.$$  

Therefore, $\nu = \nu_{\alpha''} \mathbf{v}_{\alpha''} \mathbf{v}_{\alpha''}$, due to carrier wave self-interaction: $\nu_{\alpha''} = 0$ due to the $\theta$th/2nd order harmonics. $\nu_{\theta} = \nu_{\alpha'} \mathbf{v}_{\alpha'} \mathbf{v}_{\alpha'}$ is due to the cubic term in $\tilde{\tau}$.

5. Modulational stability analysis

Lинейная оценка параметра порядка $\epsilon$ при $\epsilon = 1 / 2$.

$$\psi = \psi_0 e^{i \alpha \xi^2},$$

i.e. setting $\psi = e^{i \alpha \xi^2}$, we obtain the (perturbation) dispersion relation:

$$\nu^2 = \frac{\alpha^2}{\nu_0}.$$  

The wave will be stable $(\psi > 0)$ if the product $\nu_0 \psi^2$ is negative.

For positive $\psi > 0$, instability sets in for $k_{\psi} = \sqrt{\frac{\nu_0}{\alpha}}$.

The instability growth rate is $\frac{\partial \psi}{\partial \xi} = (\psi_0 e^{i \alpha \xi^2})$.

6. Localized envelope excitations

We finally obtain a localized modulated wave packet in the form:

$$\psi = \psi_0 e^{i \alpha \xi^2}.$$

We shall consider small deviations by taking ($\epsilon < 1$)

$$S = S_0 + \epsilon S_1 + \epsilon^2 S_2 + \ldots + \epsilon^n S_n.$$  

References

[1] For a brief review, see the Introduction and subsequent literature list in [1, 2, 4].


