I. INTRODUCTION

Ion acoustic waves in unmagnetized dusty plasmas with phase velocity between the electron and ion thermal velocities \( [V_{ji}=(2k_BT_i/m_j)^{1/2}; \ j=e,i] \) were first studied theoretically by Shukla and Silin\(^1\) and later their existence was confirmed experimentally by Barkan et al.\(^2\). Dust being a common species in a wide range of space and astrophysical plasmas such as the cometary tails and comae, interstellar clouds, Earth’s mesosphere and ionosphere, Saturn’s rings, the gossamer ring of Jupiter, and in laboratory experiments (see Refs. 3 and 4, and references therein), the study of dusty plasmas has been an important focus of much recent research. Dust particles are often of micron to submicron size, and are usually found to have negative charge (possibly as large as \( \sim 10^4 \) electron charges), depending on the environment where they occur. On the other hand, smaller dust grains may be found to be positively charged.

The presence of the dust modifies the standard ion acoustic mode, giving rise to what is termed the dust ion acoustic (DIA) wave. One of the important effects of the massive dust grains is that the associated change in the free electron density yields a corresponding change in shielding and hence in the detailed wave behavior. The linear DIA wave has an increased (reduced) phase velocity when the dust is negatively (positively) charged.

Nonlinear DIA waves have also been studied by a number of authors, e.g., Refs. 5–11. We note that most nonlinear studies used reductive perturbation theory or equivalent expansions, to study various aspects of small amplitude solitons and/or double layers, for example, Refs. 6–8, while some papers considered arbitrary amplitude DIA structures,\(^5,9–11\) using the Sagdeev pseudopotential approach.\(^12\) In the first half of the pioneering work of Bharuthram and Shukla,\(^5\) the plasma model was one of Boltzmann-distributed electrons, cold ions, and immobile negative dust. They sought positive potential solitons and found existence ranges of both normalized soliton speed \( (M) \) and amplitude \( \phi \) as a function of the fraction of negative charge residing on the dust. In the latter part of the paper they considered negative potential solitons. They did not carry out a systematic study of them, but presented Sagdeev potential curves for only two values of \( M \) and two values of mobility. In addition to studying dust-acoustic solitons, Verheest et al.,\(^9\) examined DIA solitons in a plasma model which allowed for arbitrary values of the polytropic index \( (\gamma_e) \) for the electrons, cold ions, and mobile dust. Numerical evaluation of existence domains was carried out for two values of \( \gamma_e \), viz., \( \gamma_e=1 \) (isothermal, i.e., Boltzmann) and 3/2.

While most papers that discussed DIA waves and solitons are based on electrons and ions with Maxwell–Boltzmann distributions, space plasmas are often observed to possess non-Maxwellian distributions.\(^13–19\) Such distributions may be accurately modeled by a kappa (or generalized Lorentzian) distribution.\(^13,20,21\) The three-dimensional isotropic \( \kappa \) velocity distribution is\(^13,20–23\)

\[
F_v(v) = \frac{N_0}{(\pi \kappa \theta^2)^{3/2}} \frac{\Gamma(\kappa + 1)}{\Gamma(\kappa - 1/2)} \left( 1 + \frac{v^2}{\kappa \theta^2} \right)^{-\frac{(\kappa+1)}{2}},
\]

where \( N_0 \) is the unperturbed equilibrium density; \( \theta = [(\kappa - 3/2)/\kappa]^{1/2} V_i \) is the characteristic velocity, of the order of the thermal velocity, \( V_i = (2k_BT/m)^{1/2} \); \( T \) is the characteristic kinetic temperature, which is the temperature of the equivalent Maxwellian with the same average kinetic energy;\(^21\) \( k_B \) is Boltzmann’s constant, \( \Gamma \) is the usual gamma function, and \( \kappa \) is the spectral index that determines the
hardness of the energy spectrum corresponding to the presence of excess suprathermal particles in the tail of the distribution function. Note that the characteristic velocity \( \theta \) is only defined for \( \kappa > 3/2 \), and thus when considering physical quantities derived from Eq. (1), such as the density, we shall use \( \kappa \) values that exceed 3/2. Kappa distributions reduce to Maxwellian distributions for \( \kappa \to \infty \), while for low values of \( \kappa \) they represent a “hard” spectrum with a strong non-Maxwellian tail having a power-law form at high speeds. \(^{13,20}\)

In this paper we study the behavior of and existence domains for DIA solitons that may be supported by a plasma in which the electrons are non-Maxwellian (specifically kappa-distributed). Small amplitude structures are investigated using the reductive perturbation technique, while the Sagdeev pseudopotential approach is used for arbitrary amplitude soliton studies. While most of the investigation deals with the more interesting and relevant case of negative dust, we also consider positive dust. In particular we draw attention to the occurrence of finite amplitude solitary waves at the DIA speed in a negative dust plasma and explore some of the characteristics of this phenomenon.

The structure of the paper is as follows. In Sec. I, after a brief introduction, we present the basic equations of the plasma model and derive the associated linear dispersion relation. Section II discusses the small amplitude expansions through the reductive perturbation technique, and in Sec. III we discuss the arbitrary amplitude structures following the pseudopotential/Sagdeev approach. Both positive and negative solitons are considered, and particular care is taken to investigate the region of parameter space in which solitons of both signs may be found. A brief summary and discussion rounds off the paper in Sec. IV.

A. Basic equations and linear dispersion relation

We consider a plasma with kappa-distributed electrons of temperature \( T_e \), and density \( N_{e0} \), fluid adiabatic ions of temperature \( T_i \), and density \( N_{i0} \), and cold dust particles of density \( N_d \). The charge equilibrium condition for the system is \( N_{e0} = N_{i0} + sZ_dN_{d0} \), where \( N_{i0} \) is the equilibrium density of species \( j \); \( Z_d \) is the size of the dust charge, and \( s = \pm 1 \) is the sign of the dust charge (for positive or negative dust grains). The ions are assumed to be singly charged, like protons, for example, and thus \( Z_d = 1 \) in this plasma model.

In an electrostatic potential, the \( \kappa \)-distributed electrons have normalized density \( n_e = n_e/N_{e0} \) given by \(^{24}\)

\[
n_e(\phi) = f \left( \frac{\phi}{\kappa - 3/2} \right)^{-(\kappa-1/2)},
\]

where \( \phi \) is the electrostatic potential, here normalized with respect to the electron thermal energy \( K_B T_i/e \), and the fractional electron density, \( f = N_{e0}/N_{i0} = 1 + sZ_dN_{d0}/N_{i0} \). For the more usual case of negative dust, \( f \) represents the fraction of negative charge associated with the free electrons. The density expression given above is only valid for \( \kappa > 3/2 \), and it reduces to the usual Maxwellian form \( n_e(\phi) = f \exp(\phi) \) when \( \kappa \to \infty \).

The dust and ion densities are obtained from the normalized continuity, momentum and pressure equations

\[
\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i u_i) = 0,
\]

\[
\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x} + \frac{m_i \sigma \partial p_i}{m_i} + \frac{m_i q_j \partial \phi}{m_i e} = 0,
\]

\[
\frac{\partial q_j}{\partial t} + u_j \frac{\partial q_j}{\partial x} + 3p_j \frac{\partial u_j}{\partial x} = 0,
\]

where \( q_j \) is the species charge \( (q_i = e \) and \( q_d = sZ_d e) \), and \( u_i, n_j \) and \( p_j \) are the normalized velocities, densities, and pressures, respectively. We assume the ions to be adiabatic (\( \gamma = 3 \)) and the dust to be cold (\( p_d = 0 \)). The independent variables, \( x \) and \( t \), are normalized to a mixed electron-ion Debye length \( \lambda_{DI} = (\varepsilon_0 K_B T_e/N_{e0} e^2)^{1/2} \) and the inverse ion plasma frequency \( \omega_{pi}^{-1} = (N_{i0} e^2 / e^2 m_i)^{1/2} \), respectively; the dependent variables, \( u_j, n_j, p_j \), are normalized to the ion acoustic speed \( C_s = (K_B T_e / m_i)^{1/2} \), the ion density \( N_{i0} \), and the ion pressure \( P_{i0} = N_{i0} K_B T_i \), respectively, and \( \sigma = T_i / T_e \). The variables \( u_j, n_j, p_j, \) and \( \phi \) satisfy the boundary conditions

\[
u_j \rightarrow 0; \quad n_j \rightarrow (N_{i0}/N_{d0}), \quad p_j \rightarrow 1; \quad \phi \rightarrow 0 \quad \text{as} \quad x \rightarrow \pm \infty.
\]

The fluids are coupled through Poisson’s equation

\[
\frac{\partial^2 \phi}{\partial x^2} + n_e(\phi) - n_i(\phi) + sZ_d n_d(\phi) = 0.
\]

To study the linear waves we Fourier analyze the equations in terms of normalized angular frequency \( \omega \) and wave-number \( k \) and expand them to linear order. Hence one easily obtains the linear dispersion relation

\[
1 - \frac{1}{\omega^2 - 3\sigma k^2} + \frac{f}{k^2} \left( \frac{\kappa - 1/2}{\kappa - 3/2} \right) - \frac{1}{\omega^2} z = 0,
\]

where \( z = m_i (Z_d / m_i) = sZ_d \) is the ratio of the charge-to-mass ratio of dust to that of ions (with \( Z_d = 1 \)). For the typical situation \( z \ll 1 \), Eq. (7) becomes

\[
1 - \frac{1}{\omega^2 - 3\sigma k^2} + \frac{1}{k^2 V_{s0}^2} = 0.
\]

It follows that the effective DIA speed is

\[
V_{s0} = \left( \frac{N_{i0}}{N_{e0}} \right) \left( \frac{\kappa - 3/2}{\kappa - 1/2} \right),
\]

which yields \( 1/f \) in the limit \( \kappa \to \infty \). In the long wavelength limit, \( k \ll 1 \), one obtains

\[
\omega^2 = k^2 (V_{s0}^2 + 3\sigma).
\]

Since the phase velocity \( \omega / k \) is normalized to the ion sound speed \( C_s = (K_B T_e / m_i)^{1/2} \), it follows that for Maxwellian electrons (\( \kappa \to \infty \)), immobile dust and cold ions (\( \sigma \to 0 \)), with \( V_{s0} = 1/f \), we recover the original dispersion relation, \(^1\) given in unnormalized form as \( \omega^2 = (N_{i0} / N_{e0}) k^2 C_s^2 \) with \( Z_d = 1 \).
II. SMALL AMPLITUDE SOLITONS

To study solitons moving in a stationary frame, it is convenient to work in the comoving frame, such that the plasma species flow through the stationary solitary wave structure and have an undisturbed normalized speed $M$ at $x \to \pm \infty$. It can then be shown\textsuperscript{25,26} that Eqs. (3)–(5) yield the normalized adiabatic ion density

$$n_i(\phi) = \frac{1}{2 \sqrt{3} \sigma} \left[ (M + \sqrt{3} \sigma)^2 - 2 \phi \right]^{1/2} \pm \left[ (M - \sqrt{3} \sigma)^2 - 2 \phi \right]^{1/2}. \tag{11}$$

From the boundary conditions, it follows that $n_i \to 1$ for $\phi \to 0$. Hence we have to take the minus sign in Eq. (11). In the limit $\sigma \to 0$ (cold ions), $n_i(\phi) = (1 - 2 \phi/M^2)^{1/2}$. This means that when $\phi = M^2/2$, $n_i \to \infty$, and the cold ions are infinitely compressed.

The normalized density of the cold dust particles is

$$n_d(\phi) = \frac{(f - 1)}{sz_d} (1 - 2sz \phi/M^2)^{-1/2}, \tag{12}$$

where $M$ is the Mach number (soliton speed). However, if the dust motion is not included, as the ion and electron dynamics dominate for DIA waves,

$$n_d \to N_{\phi d}/N_{\phi 0} = (f - 1)/sz_d, \tag{13}$$

since the cold dust then only provides neutralization in the background. We shall use this model when studying small amplitude solitons, but allow for dust mobility in the pseudo-potential calculations.

In the reductive perturbation method, one uses a small amplitude expansion which is cut off, and it is thus valid only for small $\phi$. This is an aspect of Korteweg–de Vries (KdV) soliton theory that is sometimes ignored. Using this approach, the electron density is obtained from

$$n_e(\phi) = f \left( 1 - \frac{\phi}{\kappa - 3/2} \right)^{-1(\kappa - 1/2)} = f + c_1 \phi + c_2 \phi^2 + c_3 \phi^3 + \cdots, \tag{14}$$

where

$$c_1 = f \left( \frac{\kappa - 1/2}{\kappa - 3/2} \right), \quad c_2 = \frac{f(\kappa - 1/2)(\kappa + 1/2)}{2!(\kappa - 3/2)^2}, \quad c_3 = \frac{f(\kappa - 1/2)(\kappa + 1/2)(\kappa + 3/2)}{3!(\kappa - 3/2)^3}, \cdots. \tag{15}$$

A word of caution is called for here in that the expansion of Eq. (2) is only valid for $\kappa > 3$. For $3/2 < \kappa \leq 3$ the higher order terms are large compared to those of lower order, and therefore cannot be neglected. Thus when one uses the reductive perturbation method for $\kappa$-distribution plasmas, the range of $\kappa$ values for which it is valid imposes an important constraint.$\textsuperscript{21}$

In addition, we here assume that the dust is cold and immobile. Using Eq. (13), Poisson’s equation becomes

$$\frac{\partial^2 \phi}{\partial x^2} + n_i(\phi) - n_e(\phi) + f - 1 = 0, \tag{16}$$

where $\xi = x - Mt$ in the wave frame, and the ion density $n_i(\phi)$ is obtained from the perturbation expansion of the ion fluid equations [Eqs. (3)–(5)].

In deriving the KdV equation we use the usual stretched coordinates \textsuperscript{3,4,6,27}$x = t - \frac{1}{2} \xi$, with $\xi = x - Mt$, where $M_0$ is the phase velocity normalized to the fixed acoustic speed in the absence of dust, and $\varepsilon$ is a smallness parameter. We obtain the KdV equation\textsuperscript{3,4,6,27}

$$\frac{\partial \phi_1}{\partial \tau} + A \frac{\partial \phi_1}{\partial \chi} + B \frac{\partial^3 \phi_1}{\partial \chi^3} = 0, \tag{17}$$

where the constants $A$ and $B$ are found from

$$A = B(12\sigma^3 + 3c_1^2 - 2c_2); \quad B = 1/(2c_1^2 M_e); \tag{18}$$

and $c_1$ and $c_2$ are defined in Eq. (15). Using the transformation $\eta = \chi - M_0 \tau = \varepsilon^{1/2} \xi$, with $\xi = x - Mt$, where $M_0$ is the normalized speed of the solitary wave in the stationary frame, and $M = M_0 + \varepsilon M_0$ is the normalized speed of the solitary waves in the laboratory frame, or simply, the Mach number, we obtain the usual solution to Eq. (17) as\textsuperscript{3,4,6,27,28}

$$\phi_1(\eta) = \left( \frac{3M_0}{A} \right) \text{sech}^2 \left( \frac{M_0}{4B} \eta \right). \tag{21}$$

Finally, transforming to the laboratory frame [with coordinates $(x, t)$] and taking $\phi \sim \varepsilon$ we get\textsuperscript{27}

$$\phi(x, t) = \left( \frac{3M_0}{A} \right) \text{sech}^2 \left( \frac{\delta M}{4B} \right) (x - Mt), \tag{22}$$

where $\delta M = \varepsilon M_0 = M - M_0$. The amplitude and width of the soliton are given by $3M_0/A$ and $(4B/\delta M)$, respectively. In particular, we note that the amplitude of a KdV soliton is zero when $M = M_0$ and is proportional to $(M - M_0)$. Since $B$ is always positive, the validity of Eq. (22) requires $\delta M > 0$, that is $M > M_0$, and therefore only supersonic DIA solitons will exist in this small amplitude model. We note also that, with $\delta M > 0$, the sign of the KdV soliton potential will depend on whether $A$ is positive or negative.

From the definition of $B$ it follows that it is always positive for $\kappa > 3$. Thus, from Eq. (18) one can, for given $\kappa$, find a critical plasma composition, i.e., a critical value of $f$, where denoted $f_c$, for which the coefficient, $A$, of the nonlinear term $(\phi \partial \phi / \partial \chi)$ in the KdV equation [Eq. (17)] is zero, and the amplitude $(3M_0/A)$ in Eq. (22) goes to infinity.

In Fig. 1 the continuous (red) curve shows the variation, with $\kappa$, of $f_c$, the solution of the equation $A = A(f; \kappa) = 0$, for fixed $\sigma$ and $z$. From the sign of $A$ one can show that positive (negative) small amplitude potential solitons are obtained for $f > f_c$ ($f < f_c$), i.e., above (below) the continuous red curve in Fig. 1. In other words, solitons with either polarity are in principle supported by the plasma model. However, for fixed values of $f$, $\kappa$, and $\sigma$, and hence of $c_1$ and $c_2$, the sign of $A$
and thus the soliton polarity, are uniquely defined, i.e., for a given plasma configuration, only a single sign of soliton potential is permitted.

This figure yields a further interesting physical result: for a plasma with, say, $f=0.4$, the figure shows that a Maxwellian-like distribution ($\kappa \gg 10$) supports positive KdV solitons ($f > f_c$), while for $\kappa = 4$, the KdV solitons would be negative ($f < f_c$). We draw attention to the fact that in the arbitrary amplitude, pseudopotential study that follows in the next section, $f_c$ will be seen to play a significant role in determining the soliton characteristics.

Critical composition. As the KdV method is invalid close to the critical composition, $f_c$, we have to turn to the modified KdV (mKdV) solution in that neighborhood. In this approach we use the stretched coordinates $\chi = \epsilon(x-M_d t)$ and $\tau = \epsilon^2 t$, and thus obtain the mKdV equation,

$$\frac{\partial \phi_1}{\partial \tau} + C \frac{\partial^2 \phi_1}{\partial \chi^2} + \frac{\partial^3 \phi_1}{\partial \chi^4} = 0,$$

in which the quadratic nonlinear term of the KdV equation is replaced by a cubic nonlinearity. The standard solution in the laboratory frame, assuming $\phi \sim \epsilon$, is

$$\phi(\xi) = \pm \left( \frac{6 \delta M}{C} \right)^{1/2} \text{sech} \left( \frac{\delta M}{B} \right)^{1/2} \xi,$$

(24)

where $\xi = x-M\tau = (\chi-M_0 \tau)/\epsilon$, $\delta M = \epsilon^2 M_0 = M-M_0$, with parameters, $M$, $M_0$, and $M_\sigma$, as well as $B$, defined as in the KdV expressions; and $C$ is found from

$$C = \frac{36 \sigma c_1^2 (2c_2 - c_1^2) + 2c_1 c_2 (5 + 2c_1)}{B^2 c_3^2 (2c_1 + 19/2) - 3c_3},$$

with $c_1$, $c_2$, and $c_3$ defined in Eq. (15). Now Eq. (24) requires that $B$, $C$, and $\delta M$ must all be of the same sign for real soliton width and potential. As $B$ is always positive, it follows that we require $\delta M > 0$ (i.e., $M > M_0$), as for the KdV solitons, and there is a further constraint, $C > 0$. In Fig. 1 the range of validity for different spectral indices $\kappa$ lies between the two dashed (light blue) curves, $C=0$.

Although the range of $f$ over which $C>0$ appears quite large in Fig. 1, the mKdV equation, like the KdV equation, applies only to small amplitude solitary waves. From Eq. (24) it is clear that small amplitude solitons require $C$ as large as possible. One can show that $C$ peaks at $f = (f_c - 0.1)$ for all $\kappa \geq 4$, with typical maxima $\mp 0.5$. Thus, in practice, a valid mKdV description is restricted to a narrower range than that given by $C>0$ in Fig. 1. From the form of Eq. (24), it follows that the polarity of mKdV solitons is not specified.

III. ARBITRARY AMPLITUDE SOLITONS

Substitution of the species’ densities from Eqs. (2), (11), and (12) in Poisson’s equation, Eq. (6), leads, after an integration, to the usual energy equation

$$\frac{1}{2} \left( \frac{\partial \phi}{\partial \xi} \right)^2 + \Psi(\phi, M) = 0,$$

(25)

where the pseudopotential $\Psi(\phi, M)$ is given by

$$\Psi(\phi, M) = \left[ 1 - \left( 1 - \frac{\phi}{\kappa - 3/2} \right)^{(\kappa/3)^2} \right]$$

$$- (1-f) \frac{M^2}{x^2} \left[ 1 - \left( 1 - \frac{2z \phi}{M^2} \right)^{1/2} \right]$$

$$+ \frac{1}{6 \sqrt{3} \sigma} \left[ \left( (M + \sqrt{3} \sigma)^2 - 2 \phi \right)^{3/2} + M^2 + \sigma, \right.$$

(26)

and the boundary conditions $\phi, \phi/\partial \xi \rightarrow 0$ as $\xi \rightarrow \pm \infty$ have been used. The three terms in this expression represent the contributions to the pseudopotential, of the $\kappa$-distributed electrons, cold mobile dust, and warm fluid ions, respectively.

In the limit $z \rightarrow 0$, the cold dust particles’ contribution to Eq. (26) becomes $(1-f) \phi$. Thus, for the case of stationary negatively charged dust particles, cold ions $[\sigma = 0; \ n_i = \frac{1}{1-2\phi/M^2}]$, and Boltzmann electrons ($\kappa \rightarrow \infty$), Eq. (26) reduces to

$$\Psi(\phi, M) = f(1-\phi) - (1-f) \phi$$

$$+ M^2 \left[ 1 - \left( 1 - 2 \phi/M^2 \right)^{1/2} \right],$$

which is essentially Eq. (8) of Bharuthram and Shukla, with $N_c = f$ and $N_d = (1-f)$ in their notation. Similarly, considering cold ions, cold mobile dust and Boltzmann electrons we recover their Eq. (19). In addition, we also observe that when $f=1$, the plasma system is completely without dust, and then for cold ions ($\sigma = 0$), we recover Eq. (19) of Saini et al., that is, the model reduces to a cold-ion/kappa-electron plasma.

In seeking solitary structures (solitons or double layers) we impose the usual conditions:

(i) $\Psi(\phi=0, M) = \Psi'(\phi=0, M) = 0$.

(ii) $\Psi''(\phi=0, M) < 0$, such that the origin is unstable.

(iii) $\Psi(\phi_0, M) = 0$ for some $M$ and the root $\phi_0 \neq 0$. 

FIG. 1. (Color online) The critical density fraction, $f_c$, where the KdV coefficient $A=0$, vs $\kappa$ for $\sigma=0.01$ and $z=0.001$ (continuous curve, red). The mKdV coefficient, $C$, is positive between the two dashed curves (light blue). The dotted (dark blue) curve, coinciding with the continuous curve for $f_c$, represents values of $f$ obtained from $\Psi''(\phi=M, \phi=0)=0$, in the arbitrary amplitude case (see next section).
(iv) $\Psi(\phi, M) < 0$ for $0 < |\phi| < |\phi_m|$ for some $M$.

(v) For double layers, in addition to (i)–(iv), $\Psi(\phi_m, M) = 0$ must hold for some $M$ and $\phi_m \neq 0$. The primes in (i), (ii), and (v) denote derivatives with respect to $\phi$.

Condition (i) is satisfied by Eq. (26). Applying the second condition, often referred to as the soliton condition, we find the constraint

$$\Psi''(M) = \frac{1}{M^2 - 3\sigma} + \left( f - 1 \right) \frac{1}{M^2} - f \left( \frac{\kappa - 1/2}{\kappa - 3/2} \right) = 0,$$  

which may be written as

$$M^2 > V_{so}^2 + 3\sigma,$$

with

$$\frac{1}{V_{so}^2} = f \left( \frac{\kappa - 1/2}{\kappa - 3/2} \right) - \left( f - 1 \right) \frac{sz}{M^2}. \tag{28}$$

The right-hand side of Eq. (28) is, of course, essentially the same as the square of the linear phase velocity found earlier [Eq. (10)], where in obtaining Eq. (10) we assumed $z \ll 1$. However, the formulation given in Eq. (28) is not fully transparent in $M$ as $V_{so}$ is itself a function of $M$. Treating Eq. (27) as a quadratic expression in $M^2$, one can instead write the constraint as

$$M^2 > M_s^2 = \frac{b}{2a} \left[ 1 + \left( 1 - \frac{4ac}{b^2} \right)^{1/2} \right], \tag{29}$$

provided $b^2 - 4ac \geq 0$ for real $M_s$, where the latter is obtained from $\Psi''(\phi=0, M) = 0$. In Eq. (29) we ignored the inappropriate negative square root. The constants $a$, $b$, and $c$ are, respectively,

$$a = f \left( \frac{\kappa - 1/2}{\kappa - 3/2} \right), \quad b = 1 + s(f - 1)z + 3\sigma a, \quad \text{and}$$

$$c = 3sz(f - 1).$$

Equation (29) represents the lower limit of the soliton existence domain in the space of $(f, M)$. From Eqs. (10), (28), and (29), it follows that $M_s$ is the true speed of DIA waves in this plasma model. For $\sigma \to 0$, $\kappa \to \infty$, we get $M_s^2 = [1 + sz(f - 1)]/f$, which reduces to $M_s^2 = 1/f$ for $z \ll 1$, as found for DIA solitons in a plasma with cold ions and polyanalastic electrons, and for ion acoustic solitons in a two-ion plasma. For an electron-ion plasma ($f=1$) one recovers the usual lower Mach number limit ($M_s = 1$).

As expressed in Eq. (29), the constraint has been written as a condition on $M$ for a fixed value of the fractional electron density, $f$. It is sometimes preferable to consider the constraint on $f$ at fixed $M$, in particular, at $M = M_s$. Then Eq. (27) can be written in the form

$$f > f_s(M) = \frac{1/(M^2 - 3\sigma) - sz/M^2}{\beta - sz/M^2}, \tag{30}$$

with $\beta = (\kappa - 1/2)/(\kappa - 3/2)$. Thus, for fixed $M$, solitons or double layers will exist for all $f > f_s(M)$.

In applying conditions (iii) and (iv) to $\Psi(\phi, M)$ we consider the constraints on the density expressions in the plasma model. Soliton regions may be bounded by a number of possible physical constraints, e.g., the occurrence of a double layer, when one of the species reaches a sonic point (for simpler models that implies infinite rarefaction or compression of the species), or a density takes on a complex value. It is usual for the density limit arising from a species of a given charge sign to lead to limitation of that sign of potential, i.e., positive particles provide positive potential limits and vice versa. However, it is easily seen from Eq. (2) that $\kappa$-distributed electrons are well behaved for all $f < 0$. Although, clearly, $n_i(\phi) \to \infty$ at $f \to \kappa - 3/2$, closer examination shows that the pseudopotential $\Psi[(\kappa - 3/2), M] \to -\infty$. Thus $\Psi$ does not satisfy the requirement for limiting the potential, viz., $\Psi[(\kappa - 3/2), M] > 0$.

It follows that in the case of negative dust, the positive ion and negative dust densities will limit the electrostatic potential for positive and negative potential solitary waves, respectively. On the other hand, for the case of positive dust, positive potential solitons will be limited typically by the ions, which have the smaller limiting potential because of their much smaller mass.

We turn next to a detailed study of the specific limiting potentials, $\phi_{\ell i}$, first that due to the ions ($\phi_{\ell i}$), and then that due to the infinite compression of dust ($\phi_{\ell d}$). The restriction on $M$ or $f$ for positive potential solitons associated with the ion density is given by $\Psi[(M - \sqrt{3}\sigma)^2/2, M] > 0$, since for $f > \phi_{\ell i} = (M - \sqrt{3}\sigma)/2$ the ion density, $n_i(\phi)$, is complex. One can easily see that $n_i(\phi)$ is also complex for $f = (M + \sqrt{3}\sigma)/2$, but, as that potential exceeds $\phi_{\ell i}$, it follows that $\Psi(\phi_{\ell i}, M) > 0$ will be the practical constraint limiting positive potential solitons. The condition $\Psi(\phi_{\ell i}, M) > 0$ leads to an upper limit on $f$, viz.,

$$f < f_{\ell i}(M) = \frac{f_{\ell d}(M)}{f_{\ell d}(M)}, \tag{31}$$

where

$$f_{\ell d}(M) = M^2 + \sigma - 4M^{3/2} \left( \frac{\sigma}{27} \right)^{1/4}$$

$$- \frac{M^2}{sz} \left[ 1 - \left( 1 - \frac{sz}{M^2} (M - \sqrt{3}\sigma)^2 \right)^{1/2} \right];$$

$$f_{\ell i}(M) = - \frac{M^2}{sz} \left[ 1 - \left( 1 - \frac{sz}{M^2} (M - \sqrt{3}\sigma)^2 \right)^{1/2} \right]$$

$$- \left[ 1 - \left( \frac{(M - \sqrt{3}\sigma)^2}{2(\kappa - 3)} \right)^{3/2-k} \right].$$

Likewise, from the dust density Eq. (12), it follows that $n_d \to \infty$ for $|\phi_{\ell d}| = [(M^2/2sz)]$. For negative dust, $s = -1$, this potential implies an upper limit on $f$ for negative potential solitons. We note in passing that, as for most practical dusty plasmas $z \ll 1$, the potential at which the dust limit occurs has a very large magnitude. Hence, for positive dust ($s = +1$), the ion limit, $\phi_{\ell i}$, leads to a constraint before $\phi_{\ell d}$ plays a role.

For the case of negative dust, the necessary condition yielding a constraint on the range of $M$ or $f$ over which
negative potential solitons can exist will be given by 
\( \Psi(\phi_{ei}, M) > 0 \), which upon using Eq. (26) leads to 
\[
f < f_{cs}(M) = \frac{f_{p}(M)}{f_{E}(M)} \tag{32}
\]
where
\[
f_{E}(M) = -\frac{M^{2}}{3\zeta} \left[ \frac{1}{1 - \frac{M^{2}}{3\zeta(2\kappa - 3)}} \right]^{3/2 - \kappa}.
\]

Equations (30)–(32) imply that in the case of negative dust, for given parameters \( \kappa, \sigma, \) and \( M \), positive potential solitons will exist in a region of parameter space \( (M,f) \) satisfying \( f_{c}(M) < f < f_{cs}(M) \) while negative solitons will be bounded by \( f_{d}(M) < f < f_{cs}(M) \). Note that the value of \( M \) corresponding to \( f_{c}(M) \) gives the lower Mach number below which no solitons exist, that is, the value of \( M \) at the soliton condition, \( M_{s} \). Likewise, the values of \( f \) associated with \( f_{d} \) and \( f_{cs} \) will give the upper Mach number limits for positive potential \( (M_{i}) \) and negative potential \( (M_{d}) \) solitons, respectively, at given \( f \).

The curves representing the lower and upper limits intersect at a critical value of \( f \), where, for positive solitons, \( f_{p} \) occurs for \( f_{c} = f_{cs} \), i.e., \( f_{p} \) is defined by \( f_{p} = f_{c}(M_{i}) \). For negative solitons, the critical value is \( f_{n} = f_{cs}(M_{d}) \). These two critical values provide cutoffs in \( f \) below (above) which, no positive (negative) solitons are supported in a plasma with negative dust grains. Similarly, in the case of positive dust, no positive solitons are supported below \( f_{p} \).

In general, it follows that for negative dust, (i) only negative solitons are observed for \( 0 < f < f_{p} \), (ii) solitons of both polarities are supported for \( f_{p} < f < f_{n} \), and (iii) only positive solitons are found for \( f > f_{n} \). When \( f \rightarrow f_{p} \), \( \phi \rightarrow \phi_{ti} \), \( \left( M - \sqrt{3}\sigma \right)^{2}/2 \) and \( n_{i}(\phi) \) becomes complex, yielding a cutoff for the existence domain. Similarly, when \( f \rightarrow f_{n} \), \( \phi \rightarrow \phi_{di} = M^{2}/2\zeta \) and \( n_{i}(\phi) \rightarrow \infty \).

In the next subsections we will evaluate Eqs. (30)–(32) for negative and positive dust, respectively, to obtain the existence domains for DIA solitons in a dusty kappa plasma, and, in particular, to find \( M_{s}, M_{i}, M_{d}, \) and the points \( f_{p} \) and \( f_{n} \).

A. Negative dust (\( s = -1 \) and \( f < 1 \))

In Fig. 2 we present existence domains of DIA solitons in the parameter space of Mach number \( (M) \) and fractional electron density \( (f) \). The domains are delineated by solutions of Eqs. (30)–(32). In the upper panel, we consider first the case studied by Bharuthram and Shukla, \(^{4}\) viz., positive solitons in a plasma composed of Maxwellian electrons \( (\kappa = \infty) \), cold ions \( (\sigma = 0) \), and immobile dust \( (\zeta = 0) \). The continuous curves essentially reproduce the results of Ref. 5. Positive solitons are supported in the domain bounded by the two curves, i.e., the lower, red (soliton existence) curve \( (f_{s} \text{ or } M_{s}) \), and the upper, blue curve \( (f_{cs} \text{ or } M_{cs}) \). Thus positive solitons may exist for \( f > f_{p} = 0.16 \), where \( f_{p} \) is the lower cutoff of \( f \) defined above. At that value of \( f \), one finds the highest Mach number at which positive solitons can be supported, \( M = 2.5 \). As expected, for \( f = 1 \) the system reduces to ion-acoustic solitons in a simple electron-ion plasma, and we observe the usual range \(^{34}\) of Mach numbers, viz., \( 1.0 < M < 1.6 \).

In this figure we also consider the effects of dust mobility, by including curves for three other values of \( z \), viz., \( z = 0.001 \) (dotted), \( 0.01 \) (dashed), and \( 0.1 \) (dot-dashed). Both the \( M_{s} \) curve and, for positive solitons, the \( M_{i} \) curve for the
mobile cases are virtually indistinguishable from the case \(z=0\). Although not shown here, we point out that for \(z>1\) (valid for a negative ion plasma, but not for dust) mobility does affect \(M\), significantly and increases the lower cutoff to \(f_p \approx 0.4\) and decreases the highest accessible value of \(M\) (at \(f=f_p\)) to \(\approx 2.2\).

For negative solitons to exist, the structure must have a speed exceeding \(M_s\), but there is effectively no upper limit in \(M\) for \(z<1\), and for the immobile dust model they can exist over the full range \(0<f<1\). In the second part of Ref. 5 they consider mobility briefly (using \(z=0.1\)), but only present examples of Sagdeev potentials for two values of \(M\). From their results it is clear that mobility has a large effect on the amplitudes of negative solitons.\(^5\) The upper panel of our Fig. 2 shows that the almost vertical (black) curves for \(f_{ed}\) or \(M_{ed}\) are affected significantly by the value of the mobility parameter, \(z\), thereby introducing a nontrivial upper cutoff in \(f\) for negative solitons. Thus the existence domains for negative solitons are found to be smaller for mobile dust grains than for immobile dust. The upper limit \(f_p\) decreases for increasing mobility from 1.0 (\(z=0\)) through 0.97 (\(z=0.001\)) and 0.92 (\(z=0.01\)) to 0.89 (\(z=0.1\)). As seen, mobility causes a small shift in relevant \(M_s\).

In the lower panel of Fig. 2 we investigate the effects of excess superthermal electrons (through choice of the parameter \(\kappa\)) on the range of existence of DIA solitons, for mobile dust \((z=0.001)\) and an ion-electron temperature ratio of \(\sigma=0.01\). We have chosen \(z=10^{-3}\) for illustrative purposes as a typical value with \(z<1\). The continuous curves represent a Maxwellian electron distribution \((\kappa=\infty)\), a typical space plasma \((\kappa=4)\) is given by dotted curves, and the dashed curves are for a strongly non-Maxwellian plasma with \(\kappa=2\). The ranges in both \(f\) and \(M\) that can support positive potential solitons are seen to decrease with increased excess superthermality (decreasing \(\kappa\)). The figure also shows that, as above, negative potential solitons exist for unbounded Mach numbers, \(M>M_s\), over a large range of \(f\), with the cutoff being virtually independent of \(\kappa\) \((f_n \approx 0.97)\). In addition we point out that, although not shown explicitly in this figure, the precise value of \(\sigma\), within the range of appropriate values, has little effect on the existence domains.

From Fig. 2 one sees that for the chosen parameter values, both positive and negative potential solitons are supported in the range \((0.21, 0.97)\) in a Maxwellian plasma. For \(\kappa=4\), the range is reduced to \((0.31, 0.97)\) and in a strongly non-Maxwellian plasma with \(\kappa=2\), the range supporting both polarities is \((0.43, 0.97)\). Thus decreasing the spectral index \(\kappa\) from a Maxwellian to a hard spectrum has a significant effect on the range of \(f\) (through \(f_p\)) and of \(M\), over which solitons of both polarities may exist.

We shall show below that the critical values of the fractional electron density \(f\) that have been introduced above, viz., \(f_p\), \(f_c\), and \(f_n\), play an important role in providing a better understanding of the soliton characteristics in a three-component plasma for which there is a range in \(f\) in which both positive and negative potential solitons are supported. We shall later consider in Fig. 4 a plasma with \(\kappa=2\), \(\sigma=0.01\), and \(z=0.001\). From \(\Lambda(f_c)=0\) or \(\Psi''(\phi=0,M_s,f_c)=0\), and Eqs. (30)–(32), one finds that for these parameters \(f_c \approx 0.523\), \(f_p \approx 0.428\), and \(f_n \approx 0.97\). We show the values \(f_p, f_c, f_n\) explicitly in the existence diagram for this case.

We also considered the effect of the normalized ion temperature \((\sigma)\) on the existence domain of DIA solitons. It is found that for warmer ions, the range in \((f,M)\) space over which solitons can be obtained is increased slightly. Although not shown here, it is found that, varying the spectral index \(\kappa\) with fixed Mach number \(M\) and \(f\), the soliton amplitude \(\phi_m\) increases with decreasing \(\kappa\), that is, the more superthermal particles are in the high energy tail of the distribution, the higher the amplitude of the associated solitons at fixed soliton speed. However, as \(\kappa\) is decreased, the minimum soliton speed \(M_s\) is also decreased, and so the speed relative to the DIA speed is increased, thus explaining the higher amplitude.\(^{29}\) Hence in Fig. 3, we prefer to show the effect of \(\kappa\) on the soliton amplitude as a function of the soliton speed normalized to the true acoustic speed \((M/M_s)\).

In the upper left panel of Fig. 3, we consider positive potential solitons in plasmas with different \(\kappa\) values, for the case \(f=1\), i.e., for ion-acoustic solitons in a pure electron-ion plasma, as discussed in detail previously.\(^{29}\) However, whereas in the latter paper the plot was made against \(M-M_s\), here we used \(M/M_s\). As found earlier, the \(\phi_m-M\) curves decrease monotonically with decreasing \(\kappa\), i.e., with increasing excess superthermal electrons. The upper limit in \(M\) for positive solitons also decreases with decreasing \(\kappa\), as found for IA solitons in Ref. 29. While for \(\kappa=2\) one has small amplitude solitons over the full existence range, they go beyond the KdV range for higher \(\kappa\).\(^{29}\) The upper right panel of Fig. 3 shows that when some dust is included \((f=0.9)\) the results for positive solitons are very similar to those for IA solitons, but with slightly larger amplitudes.
When we consider a case with a larger dust charge density ($f=0.5$), the amplitudes increase even further, although they are still of order one in normalized magnitude, as seen in the lower left panel of Fig. 3. In addition, however, two important changes are observed. First, the curves no longer vary monotonically—they cross each other. Secondly, we find the surprising result that for $\kappa=2$ the amplitude $\phi_m$ is nonzero for $M/M_s=1$, i.e., a nonzero soliton exists at the acoustic speed, something that goes completely against KdV theory for small amplitude solitary waves.

Finally, in the lower right panel of Fig. 3 we show negative potential solitons for $f=0.9$ (i.e., the companion figure to the upper right panel). In this case, we find that the curves again vary monotonically with $\kappa$, but for all $\kappa$ negative solitons with finite amplitude are found at the acoustic speed. These solitons are of orders of magnitude larger than the positive solitons for the same plasma configuration, e.g., at the lowest Mach number supporting solitons, $M/M_s=1$, solitons have amplitudes $|\phi_m|=45$ ($\kappa=2$), 108 ($\kappa=4$), 138 ($\kappa=10$), and 156 ($\kappa=\infty$, i.e., Maxwellian), respectively. Furthermore, as negative solitons are effectively unbounded in Mach number, increasing $M$ can yield extremely large amplitudes. Large amplitude negative solitons were also reported in Ref. 5 with Maxwellian electrons, $z=0.1$, $\sigma=0$, and $f=0.7$ (see their Fig. 4). However, they did not examine the peculiar behavior at the lowest Mach numbers.

To examine further these large amplitude negative potential solitons, we carried out calculations for different parameters, as shown in Table I. For comparison, the results of Ref. 5 for $M=1.75$ are incorporated in the table and marked with an asterisk. The two sets of calculations are consistent with one another; the amplitudes are virtually independent of the normalized ion temperature, $\sigma$, but they do depend strongly on mobility, particularly over the range $0.01 \leq z \leq 0.1$.

Using a specific case study, viz., a plasma with $\kappa=2$, $\sigma=0.01$, and $z=0.001$, we next examine in Fig. 4 the role of $f_c$ and its neighborhood. Specifically, we consider the dependence of soliton amplitude on the Mach number (in terms of $M/M_s$) for $f$ in the range $(f_p,f_c)$. We recall that for these parameter values, solitons of both polarities are found in the range $(0.43, 0.97)$, while $f_c=0.52$.

In the upper left panel we present the amplitudes of positive solitons as a function of $M/M_s$ for some values of $f$. First, we note that for $f_c=f<f_n$ (for instance $f=f_c=0.55$ and 0.6), the amplitudes of positive solitons vanish for $M/M_s=1$, and they increase monotonically as $f$ approaches $f_c$. In addition, the range of $M/M_s$ that supports solitons becomes narrower. Turning next to $f<f_c$ (e.g., $f=0.48$ and 0.5), we see that, although the trends of increasing $\phi_m$ and decreasing range in $M$, with decreasing $f$, persist, one now finds that the amplitude of positive solitons is not zero at $M=M_s$.

In the upper right panel we present similar curves for negative solitons for the same values of $f$ (please note the change of scale of $\phi_m$). Again the amplitudes vary monotonically with $f$, but the solitons have zero amplitude for $f_c<f\leq f_n$ at $M=M_s$, while in the range $f_c<f<f_n$ amplitudes are nonzero at $M=M_s$. The negative solitons in general have larger amplitudes than their positive counterparts. In the middle left panel of Fig. 4 we show the pseudopotential plot for a case with nonzero positive amplitude at $M=M_s$, viz., with $f=0.5$, $M=M_s$, $\phi_m=0.09$. Although we find that $\phi_m \neq 0$ at the DIA speed for this example in the range $f_c<f<f_n$, we see that the usual requirement of a maximum of the pseudopotential at the origin [$\Psi''(\phi=0,M_s)<0$] is not satisfied. Instead, the function $\Psi(\phi,M)$ has a point of inflexion at the origin, with $\Psi''(\phi=0,M_s)=0$, while the convexity requirement at the origin is provided by the third derivative, $\Psi'''(\phi=0,M_s)<0$. We point out that a finite amplitude soliton at the acoustic speed has recently been found in a study of dust-acoustic solitons in another three-component plasma, viz., one composed of negatively charged fluid dust and two positive ion species, a cooler Boltzmann and a hotter non-thermal Cairns distribution. In that case, too, it was found to occur in conjunction with a point of inflexion in the pseudopotential at the origin, rather than a maximum, as is normally required for a soliton.

We emphasize that these structures obtained at the acoustic speed are indeed typical solitons, as may be seen from the potential profile in the middle right panel, and also reported recently by Verheest and Hellberg. This interesting result implies that the usual convexity requirement at the origin [$\Psi''(\phi=0,M_s)<0$] is a necessary but not a sufficient condition for the existence of solitons, specifically for models that support existence of solitons of both polarities. Furthermore, these finite amplitude solitary waves cannot be found by a KdV approach, as the latter solitons have $\phi_m=0$ for $M=M_s$, as discussed in Sec. II.

In the lower left panel of Fig. 4 we show the pseudopotential for a marginally subacoustic structure speed ($M=M_s-0.0001$). Clearly, the positive pulse seen in the middle left panel disappears for $M<M_s$, however small the reduction below the DIA speed: the pseudopotential has no well, and no soliton is found. On the other hand, for $M>M_s$ ($M=M_s+0.0023$) one sees that the positive soliton has a slightly increased amplitude, while a smaller amplitude negative soliton (which vanished at $M=M_s$) is observed.

This phenomenon is explored further in Fig. 5, which shows soliton amplitudes at the DIA speed, $M_s$, as a function of $f$, in the range $(f_p,f_n)$, for different values of $\kappa$. Clearly the points of intersection with the line $\phi=0$ define critical values of $f$; they occur where $\Psi''(\phi=0,M=M_s,\kappa)=0$. These values are plotted as a dotted curve in Fig. 1 and are seen to be the same as the value $f_c$ defined in Sec. II as the...
solution to the equation $A(f; \kappa)=0$. Here $f_c \approx 0.523$, 0.419, 0.365, and 0.329 for $\kappa=2$, 4, 10, and $\infty$, respectively. At $f_c$, the amplitudes of both polarities of soliton vanish at the DIA speed, and, as we have seen in Sec. II, KdV theory has to be replaced by the mKdV ansatz. As seen in Fig. 5, for each value of $\kappa$, positive potential solitons have $\Phi_p \neq 0$ at $M=M_s$ for $f_r < f < f_c$, increasing with $|f-f_r|$ as one approaches $f_r$, but (not shown in figure) we find that the amplitudes vanish at the acoustic speed for $f > f_c$. For $M=M_s$, however, these solitons have finite amplitude. On the other hand, negative solitons have zero amplitude at $M_s$ for $f < f_c$ (not shown in figure; again, with nonzero amplitudes for $M > M_s$), and take on finite values at $M_s$ for $f_r < f < f_n$, increasing with $|f-f_r|$ as $f \to f_n$. The largest positive and negative soliton amplitudes at the acoustic speed occur for $f=f_p$ and $f=f_n$, respectively.

FIG. 4. (Color online) Upper panel: plot of $\phi_m$ vs $M/M_s$ for positive (left) and negative (right) solitons, for different $f$ in the range $(f_p; f_n)$. Note the two different scales for $\phi_m$. Parameter values: $\kappa=2$, $\sigma=0.01$, and $z=0.001$. Middle panel: pseudopotential plot at $f=0.5$, $M=M_s=0.835$, and the associated potential profile of the soliton. Lower panel: pseudopotential plots at $M_s-0.0001$ (left) and $M_s+0.0023$ (right).
In summary, as $f$ is varied, the solitons of either polarity switch at $f_c$ from “KdV-like” behavior (vanishing at $M=M_*$), to a “non-KdV-like” form with $\phi_{td} \neq 0$ at the DIA speed. Equivalently, as $f$ is increased through $f_c$, the “KdV-like” solitons change sign from negative to positive, while the “non-KdV-like” structures switch from positive to negative potential. Of course, negative solitons are effectively unbounded in $M$ and can thus have very large amplitudes, but in Fig. 5 we have shown the amplitudes only up to 2, although $|\phi_{td}|$ lies in the range 150–500.

B. Positive dust ($s=+1$ and $f>1$)

We have already seen that in the case of positive dust grains, positive solitons are limited by ion compression (as $\phi_{ti}<\phi_{td}$), while negative solitons, if they exist, would be limited by the occurrence of double layers, if the latter are supported by this plasma model. However, the double layer requirements [$\Psi(\phi_{tn},M)=\Psi'(\phi_{tn},M)=0$] are not met for this model. Both $\Psi$ and $\Psi' \rightarrow -\infty$ as $\phi \rightarrow -\infty$, so no double layers can form. This observation agrees, for the Maxwellian case, with earlier work. More insight into the existence of negative solitons can be obtained from the sign of $\Psi''(\phi=0; M=M_*)$. We saw in Sec. II that small amplitude negative solitons can be obtained only for $f<f_c$. As seen in Fig. 1, $f_c<1$ for all $\kappa$. This means that for positive dust ($f>1$) only one sign of potential can be supported. Thus only positive potential solitons can occur in dusty plasmas with positive dust, kappa electrons, and fluid ions. The existence domains for positive solitons are shown in Fig. 6 (left panel) for $\kappa=2$, 4, and $\infty$, and (for $\kappa=2$) over an extended range in positive dust charge density in the right panel. We see that the existence domains are extensions of those seen for $f<1$, and that they appear similar to each other, but for decreasing $\kappa$ both the typical values of $M$ and the accessible ranges in $M$ are reduced.

IV. DISCUSSION AND SUMMARY

Using the pseudopotential approach, we have studied arbitrary amplitude DIA solitons in a plasma of positive ions, $\kappa$-distributed electrons, and charged dust grains. This represents a considerable extension of the work of Ref. 5.

For the case of negative dust, we have shown that for all $\kappa$ the model supports both positive and negative potential solitons, where the Mach number for positive (negative) potential solitons is limited from above by the condition at which the ion density becomes complex (the dust is infinitely compressed). This agrees with the analysis of Verheest et al. for polytropic electrons. We prefer not to use the commonly used word “coexist” in this context, as coexistence seems to imply that in a specific plasma configuration, both polarities can exist at the same time, whereas in fact only one will occur, and which of the two polarities will be observed depends on details of the initial disturbance.

Positive potential DIA solitons experience a low-$f$ cutoff ($f_p$) which decreases with increasing $\kappa$ (i.e., with a decrease in excess superthermal particles), and hence this increases the range in $(f,M)$ space over which positive solitons exist. Allowing for finite dust grain mobility has little or no effect on the existence domain for positive solitons, while the ion temperature (through $\sigma$) has a weak effect, increasing the size of the existence domain as it is increased. Negative potential solitons do not exist above a $\kappa$-independent cutoff $f_n \sim 0.9–1$, the exact value of which depends significantly on the magnitude of the dust mobility factor $z=z_d m_d/M_d$. They are effectively not subject to an upper limit in $M$ as $z \ll 1$ implies that $\phi_{td} \gg 1$, and thus negative solitons may be very large.

A surprising result is that over the range of fractional electron density $f$ in which solitons of both polarities are supported, finite amplitude solitary structures occur even at the DIA speed—behavior which contradicts KdV theory. Recently a similar result was found in another three-component plasma, where, as here, the phenomenon is associated with a point of inflexion of the pseudopotential at $\phi=0$ and $M=M_*$, rather than the usual maximum. The sign of
$\Psi''(\phi=0; M=M_s; f)$ then designates the polarity of the KdV-like soliton that vanishes at $M=M_s$.

A critical role is played by $f_c$, the value of $f$ at which the KdV coefficient $A=0$, which also satisfies the constraint $\Psi''(\phi=0; M=M_s; f_c)=0$. In particular, as $f$ is varied, solitons of each polarity switch at $f=f_c$ from a “KdV-like” form to “non-KdV-like” behavior. For $f_p < f < f_c$, positive solitons at $M=M_s$ have finite amplitude, increasing in size with $|f-f_c|$ as $f$ approaches $f_p$, while negative solitons have zero magnitude at $M=M_s$, as expected from KdV theory. This situation reverses in polarity for solitons found for $f_c < f < f_p$.

On the other hand, in a plasma with positive dust grains, only positive potential (“KdV-like”) solitons are supported by the plasma model, with the upper limit on $M$ provided by infinite compression of the ions. The Maxwellian case agrees with earlier results. Decreasing $\kappa$ leads to small reductions in both the accessible $M$ and the existence range in $M$. The dusty plasma model with positive dust is similar to a two component ion-electron plasma, with modifications to the dynamics due to the presence of weakly mobile dust. The results are reminiscent of those found for ion acoustic solitons in a two-ion plasma, but for a much heavier second “positive ion.”

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