Nonlinear excitations in Debye bi-crystals

B Farokhi

Department of Physics, Arak-University, Arak, Iran

E-mail: b-farokhi@araku.ac.ir

I Kourakis and P K Shukla

Ruhr-Universität Bochum, Theoretische Physik IV, Fakultät für Physik und Astronomie,

D-44780 Bochum, Germany.

Abstract

The nonlinear dynamics of longitudinal dust lattice waves (LDLWs) propagating in a dusty plasma bi-crystal is investigated. A “diatomic”-like one-dimensional dust lattice configuration is considered, consisting of two distinct dust grain species with different charges and masses. Two different frequency dispersion modes are obtained in the linear limit, namely an optical and an acoustic wave dispersion branch. Nonlinear solitary wave solutions are shown to exist in both branches, by considering the continuum limit for lattice excitations in different nonlinear potential regimes. For this purpose, a Generalized Boussinesq (GBq) and an extended Korteweg de Vries (EKdV) equation(s) are derived, for the acoustic mode excitations, and their exact soliton solutions are provided and compared. For the optic mode, a nonlinear Schrödinger-type equation is obtained, which is shown to possess bright- (dark-) type envelope soliton solutions in the long (short, respectively) wavelength range. Optic-type longitudinal wavepackets are shown to be generally unstable in the continuum limit, though this is shown not to be the rule in the general (discrete) case.

PACS Numbers: 52.27.Lw, 63.20.-e, 63.20.Ry

I. INTRODUCTION

Dusty (or complex) plasmas¹ (DP) have attracted a great deal of interest recently. A unique characteristic of DP is the formation of strongly-coupled crystalline configurations, due to the high electric charge acquired by massive dust particulates (“grains”) and the high densities and low temperatures encountered in such plasmas. Crystal formation and dynamics have been studied in various discharge experiments¹⁻³, where charged dust particles were created by injecting artificially manufactured micron-sized neutral particulates, which subsequently acquire a high charge once the gas discharge is switched on. Various two- and three- dimensional crystalline configurations were formed in such experiments and subsequently studied theoretically¹⁻⁴. This research was naturally extended by experimentally creating^{5,6} dust crystals in one-dimension and modeling their dynamics⁷⁻¹³, in view of technological applications.

The linear dust-lattice wave (DLW) profile in a one-dimensional (1D) dust monolayer consisting of identical dust grains includes an acoustic longitudinal dust-lattice (LDL) mode^{1,7} (in the x -direction), and a transverse mode^{1,8} (off-plane, along the direction of the force of gravity, say z); interestingly, the latter features an inverse optic-like (backward wave) dispersion relation. The generic characteristics of these modes depend on the nature of interactions among charged dust grains, which are most often assumed to be of screened Debye-type. However, this description is qualitatively invalidated when charge polarization (potential “dressing”) effects are taken into account^{9,10}. Electrostatic dressing effects are also taken into account in the model that will be presented in the following.

The nonlinear dynamics of a 1D dust crystal were extensively investigated from first principles in Ref. 11, which extended a previously introduced longitudinal dust-lattice soliton concept⁷ to a Boussinesq and extended Korteweg-de Vries (eKdV) type description. Envelope soliton modes were also studied in such a crystal, with respect to the longitudinal¹² and transverse¹³ degrees of freedom.

Those studies relied on a nonlinear Schrödinger (NLS)–type model of modulated dust-lattice wave packets.

Although dusty plasmas have usually been considered to contain negatively charged dust particulates¹, positively charged dust may also occur, under experimental conditions investigated, e.g., in Ref. 14, as well as in cosmic environments¹⁵. A number of recent theoretical and experimental investigations, as well as cosmic observations, have focused on the dynamics of binary dusty plasmas, consisting of two different dust species, which bear different masses and/or opposite charge polarity^{16,17}. Recent experiments involving distinct groups of dust grains are reviewed in Ref. 16. Two different sizes of dust grains are used to determine complex plasma boundaries with respect to dust void formation in gas discharge laboratory experiments¹⁷, which is known to be a precursor to dust crystallization. The formation of binary dusty plasma crystalline configurations, though admittedly not yet observed in practice, is naturally anticipated as the next step in this direction. Inspired by these experiments, a one-dimensional dust bi-crystal, i.e. a dust grain array consisting of two superposed sub-lattices of dust grains (mimicking the structure of a diatomic array in solid state physics), was studied in Ref. 18. The linear spectrum of such a bi-crystal was shown to possess an acoustic and an optic mode, and such will be the case in our model below as well. Note that the linear spectrum in such a bi-crystal is different from the single 1D layer (“monoatomic” like) case; in particular, the optic *longitudinal* mode found in Ref. 18 (though of similar, backward wave dispersion structure) is clearly to be distinguished from the optic-like *transverse* mode associated with vertical (off-plane) motion in dust monolayer⁸. The methodology employed here adopts know-how from the nonlinear description presented in Ref. 18, which in turn relied on generic soliton models (of the Boussinesq and Korteweg-de Vries type) of collective excitations associated with longitudinal dust grain displacements¹¹.

In this paper, we aim at studying longitudinal motion in a one-dimensional horizontal chain consisting of two types of dust grains, specifically bearing different (negative) electric charge and different masses; see Fig. 1. We assume that the two different species coexist at the same levitation height, where electric forces balance gravity (i.e., the condition $QE = Mg$ is independently met by both dust species; here Q and M denote the dust charge and mass, respectively, while E is the electric field and g is the acceleration of gravity on Earth; recall that the charge may be a function of the levitation coordinate¹⁹ and that a nonlinear dependence of the dust charge on dust mass exists²⁰. The outline of this paper goes as follows. In Section II, we derive a set of discrete and, separately, (approximate) continuum equations of motion for dust grains. The nonlinear aspects of acoustic and optic-like motion are analyzed in Sections III and IV. Our results are summarized and discussed in Section V.

II. EQUATIONS OF MOTION

Let us consider a one-dimensional horizontal chain (assumed infinite, for simplicity) consisting of different negative charge dust grains Q_1 (Q_2) and different mass M_1 (M_2 , respectively), modeled as point charges. Vertical force equilibrium is ensured by gravity and electric/magnetic forces. Transverse dust grain motion is assumed to be frozen, so only longitudinal displacement, say $\delta x_n = x_n - na$ (where $n \in N$), is permitted in this simplified model (see Figure 1).

The electrostatic binary interaction force $F(r)$ exerted on two grains situated at a distance r is derived from an electrostatic potential function $U(r)$, viz. $F(r) = -\partial U(r) / \partial r$. We may expand the potential energy around equilibrium at $r = r_0$,

$$U(r) = U(r_0) + (r - r_0) \left. \frac{\partial U}{\partial r} \right|_{r=r_0} + \frac{1}{2} (r - r_0)^2 \left. \frac{\partial^2 U}{\partial r^2} \right|_{r=r_0} + \frac{1}{6} (r - r_0)^3 \left. \frac{\partial^3 U}{\partial r^3} \right|_{r=r_0} + \frac{1}{24} (r - r_0)^4 \left. \frac{\partial^4 U}{\partial r^4} \right|_{r=r_0}, \quad (1)$$

or, upon defining $G_1 = (\partial^2 U / \partial r^2)|_{r=r_0}$, $G_2 = (\partial^3 U / \partial r^3)|_{r=r_0} / 2$ and $G_3 = (\partial^4 U / \partial r^4)|_{r=r_0} / 6$,

$$U(r) \cong \frac{1}{2} G_1 (r - r_0)^2 + \frac{1}{3} G_2 (r - r_0)^3 + \frac{1}{4} G_3 (r - r_0)^4. \quad (2)$$

We have calculated the polynomial coefficients G_1 , G_2 , and G_3 for the "dressed" potential energy

$$U_D(r) = \frac{Q_1 Q_2}{4\pi\epsilon_0} \left(1 - \frac{r}{2\lambda_D}\right) \frac{\exp[-r/\lambda_D]}{r}.$$

For *first neighbor interactions* (FNI), we have obtained

$$G_1(\kappa) = 2G \frac{1 + \kappa + \kappa^2/2 - \kappa^3/4}{\kappa^3} \exp(-\kappa), \quad (3)$$

$$G_2(\kappa) = -3 \frac{G}{\lambda_D} \frac{1 + \kappa + \kappa^2/2 + \kappa^3/6 - \kappa^4/12}{\kappa^4} \exp(-\kappa), \quad (4)$$

$$G_3(\kappa) = 4 \frac{G}{\lambda_D^2} \frac{1 + \kappa + \kappa^2/2 + \kappa^3/6 + \kappa^4/24 - \kappa^5/48}{\kappa^5} \exp(-\kappa), \quad (5)$$

where $G = Q_1 Q_2 / 4\pi\epsilon_0 \lambda_D^3$ is a characteristic quantity, which physically corresponds to the “spring

constant” $G = \frac{1}{2} \mu \omega_{LDL,0}^2$; here, μ denotes the *reduced mass* $\mu = M_1 M_2 / (M_1 + M_2)$ and $\omega_{LDL,0}$ is a

characteristic LDL vibration frequency (in principle provided by the experiment). We have also defined the lattice parameter $\kappa = r_0 / \lambda_D$ (a positive real number, practically taking values $\kappa \cong 1 - 1.5$ in real discharge experiments²). Note that, upon canceling the negative contribution (i.e., the last term) in each numerator in (3)-(5), one readily obtains the corresponding quantities for the ordinary screened Debye potential (i.e. ignoring polarization). See that, ignoring polarization (i.e. for pure Debye-type interactions), G_1 and G_3 are positive quantities, while G_2 (representing the cubic contribution to the electrostatic potential nonlinearity) is negative. The interaction potential energy presents a minimum (either with or without polarization being taken into account), so particles perform (nonlinear, in general) vibrations around the equilibrium positions.

We shall also make use of the *second neighbor interaction* (SNI) quantities, $G_1' = (\partial^2 U / \partial r^2) |_{r=2r_0}$,

$G_2' = (\partial^3 U / \partial r^3) |_{r=2r_0} / 2$ and $G_3' = (\partial^4 U / \partial r^4) |_{r=2r_0} / 6$, which read (for species 1):

$$G_1'(\kappa) = 2G' \frac{1 + 2\kappa + (2\kappa)^2 / 2 - (2\kappa)^3 / 4}{(2\kappa)^3} \exp(-2\kappa), \quad (3)'$$

$$G_2'(\kappa) = -3 \frac{G'}{\lambda_D} \frac{1 + 2\kappa + (2\kappa)^2 / 2 + (2\kappa)^3 / 6 - (2\kappa)^4 / 12}{(2\kappa)^4} \exp(-2\kappa), \quad (4)'$$

$$G_3'(\kappa) = 4 \frac{G'}{\lambda_D^2} \frac{1 + 2\kappa + (2\kappa)^2 / 2 + (2\kappa)^3 / 6 + (2\kappa)^4 / 24 - (2\kappa)^5 / 48}{(2\kappa)^5} \exp(-2\kappa), \quad (5)'$$

where $G' = Q_1^2 / 4\pi\epsilon_0\lambda_D^3$. The latter expressions hold when interactions among dust grains of species 1 are considered [to be employed in Eq. of motion (6a) below]. An analogous set of expressions (here omitted) hold when SNI among species 2 grains are considered; the corresponding quantities $G_1''(\kappa)$, $G_2''(\kappa)$ and $G_3''(\kappa)$ [appearing in Eq. of motion (6b) below] are obtained from (3)'-(5)', upon a trivial change $G' \rightarrow G'' = Q_2^2 / 4\pi\epsilon_0\lambda_D^3$ in the right-hand side.

The equations of motion for the displacements u_n and w_n of grains 1 and 2 in the n -th cell are given by

$$\begin{aligned} M_1 \ddot{u}_n = & G_1[(w_n - u_n) - (u_n - w_{n-1})] + G_1'[(u_{n+1} - u_n) - (u_n - u_{n-1})] \\ & + G_2[(w_n - u_n)^2 - (u_n - w_{n-1})^2] + G_2'[(u_{n+1} - u_n)^2 - (u_n - u_{n-1})^2] \\ & + G_3[(w_n - u_n)^3 - (u_n - w_{n-1})^3] + G_3'[(u_{n+1} - u_n)^3 - (u_n - u_{n-1})^3] \end{aligned} \quad (6a)$$

$$\begin{aligned} M_2 \ddot{w}_n = & G_1[(u_{n+1} - w_n) - (w_n - u_n)] + G_1''[(w_{n+1} - w_n) - (w_n - w_{n-1})] \\ & + G_2[(u_{n+1} - w_n)^2 - (w_n - u_n)^2] + G_2''[(w_{n+1} - w_n)^2 - (w_n - w_{n-1})^2] \\ & + G_3[(u_{n+1} - w_n)^3 - (w_n - u_n)^3] + G_3''[(w_{n+1} - w_n)^3 - (w_n - w_{n-1})^3] \end{aligned} \quad (6b)$$

where all coefficients G_i (for FNI) and $\{G_i', G_i''\}$ (for SNI) (here $i=1,2,3$) were defined above.

A. Discrete linear dispersion relation

Keeping only the linear contributions in the discrete equations of motion (6a,b) (i.e. the first line in each equation), and assuming a harmonic solution of the type $u_n = \tilde{u}_n \exp[i(2knr_0 - \omega t)] + c.c.$ and $w_n = \tilde{w}_n \exp[i(2knr_0 - \omega t)] + c.c.$ (where *c.c.* stands for the complex conjugate of the preceding quantity everywhere), we obtain the dispersion relation

$$\left(\frac{M_2 \omega^2}{2G_1} - 1 - \frac{2G_1''}{G_1} \sin^2(kr_0) \right) \left(\frac{M_1 \omega^2}{2G_1} - 1 - \frac{2G_1'}{G_1} \sin^2(kr_0) \right) = \cos^2(kr_0) \quad (7)$$

Solving for ω^2 , we obtain two exact solutions, in the form

$$\omega_{\pm}^2 = \frac{G_1}{\mu} \left[\eta \pm \sqrt{\eta^2 - \frac{4\mu^2}{M_1 M_2} \left(1 + \frac{2G_1'}{G_1} + \frac{2G_1''}{G_1} + \frac{4G_1' G_1''}{G_1^2} \sin^2(kr_0) \right) \sin^2(kr_0)} \right] \quad (8)$$

where μ denotes the *reduced mass* (defined above) and

$$\eta = 1 + \frac{2\mu}{G_1} \left(\frac{G_1'}{M_1} + \frac{G_1''}{M_2} \right) \sin^2(kr_0).$$

These dispersion curves determine an acoustic mode (lower branch) $\omega = \omega_-$ [see that $\omega_-(k=0) = 0$], and an optic mode (upper branch) $\omega = \omega_+$, for which $\omega_+(k=0) = \sqrt{2G_1/\mu} \equiv \omega_0$ (retain, for later reference, the definition of the gap frequency ω_0).

Expanding near $k=0$, relation (8) gives the approximate (long wavelength) expressions

$$\omega_-^2 \approx c_1^2 k^2 \quad \text{and} \quad \omega_+^2 \approx \omega_0^2 - c_2^2 k^2,$$

where we have defined the characteristic velocities $c_1^2 = 2(G_1 + 2G_1' + 2G_1'')r_0^2/(M_1 + M_2)$ and

$c_2^2 = c_1^2 - 4r_0^2(G_1'/M_1 + G_1''/M_2)$. Note that the optical mode is characterized by inverse dispersion, i.e.

by a negative group velocity $v_{g,+} = d\omega_+/dk \approx -c_2^2 k / \omega_+$. This is manifested by the negative slope in the

upper curves in Fig. 2. Note that $c_2 < c_1$ due to second-neighbor interactions (only; in fact, the two characteristic velocities coincide for first-neighbor interactions¹⁸). The dispersion curve is depicted in figure 2.

B. Continuum approximation

To solve Eqs. (6a,b) in the continuum limit, i.e. assuming a weak variation of the function's value in space, we shall consider the displacements to be a continuous function of the space coordinate x , viz.

$u_n(t) \rightarrow u(x, t)$. We may therefore use the following expansions

$$u_{n\pm 1} \approx u_x \pm 2r_0 u_x + 2r_0^2 u_{xx} \pm \frac{4}{3} r_0^3 u_{xxx} + \frac{2}{3} r_0^4 u_{xxxx} \pm \dots \quad (9a)$$

and

$$w_n = a \left(u + \sigma_1 r_0 u_x + \sigma_2 \frac{r_0^2}{2} u_{xx} + \sigma_3 \frac{r_0^3}{6} u_{xxx} + \sigma_4 \frac{r_0^4}{24} u_{xxxx} + \dots \right) \quad (9b)$$

where r_0 stands for lattice separation which is assumed to be infinitesimally small in the long wave length limit, so that $x = nr_0$ behaves as a continuous variable. The subscripts here (or primes, later) henceforth denote partial differentiation with respect to x , while dots will denote differentiation with respect to time. Note that the former expansion (9a) is plainly a Taylor expansion around an equilibrium periodicity (spacing) $2r_0$, while the latter one (9b) is a compatibility ansatz, allowing one to decouple the set of motion equations (see in Ref. 18 and references therein): the coefficients $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ and a are (real) constant quantities, to be determined by compatibility requirements. Note that all of these coefficients should tend to unity in the equal-mass limit (for $M_2/M_1 = 1$), so that (9b) then also becomes a simple Taylor expansion (around an equilibrium spacing r_0).

III. ACOUSTIC NONLINEAR SOLITARY WAVES

For $a = 1$ (acoustic wave), and retaining orders up to r_0^4 , we obtain

$$\sigma_1 = 1,$$

$$\sigma_2 = \frac{2M_1}{M_1 + M_2} + \frac{4(G_1''M_1 - G_1'M_2)}{G_1(M_1 + M_2)},$$

$$\sigma_3 = \frac{4M_1 - 2M_2}{M_1 + M_2} + \frac{12(G_1''M_1 - G_1'M_2)}{G_1(M_1 + M_2)}$$

$$\sigma_4 = \frac{8M_1 - 6\sigma_2^2 M_2}{M_1 + M_2} + \frac{8(G_1''M_1 - G_1'M_2)(3\sigma_2 + 2)}{G_1(M_1 + M_2)}.$$

Note that all of these quantities are equal one in the equal mass (“monoatomic” chain) like.

Taking these definitions into account, the equations of motion (6-a,b) reduce to the equation

$$\ddot{u} - c_o^2 u_{xx} = p_o u_x u_{xx} + q_o (u_x)^2 u_{xx} + h_o u_{xxxx} \quad (10)$$

where we have defined the characteristic velocity (cf. above)

$$c_o^2 = r_0^2 (\sigma_2 G_1 + 4G_1') / M_1 = c_1^2,$$

and the coefficients

$$p_o = 2(\sigma_2 G_2 + 8G_2') r_0^3 / M_1, \quad q_o = 3(\sigma_2 G_3 + 16 G_3') r_0^4 / M_1, \quad h_o = (G_1 \sigma_4 + 16G_1') r_0^4 / 12M_1.$$

A. Boussinesq theory

Defining $y = \partial u / \partial x$, it is straightforward to obtain from (10) the *generalized Boussinesq (GBq) equation*

$$\ddot{y} - c_o^2 y_{xx} = p(y^2)_{xx} + q(y^3)_{xx} + h_o y_{xxxx} \quad (11)$$

where $q = q_o / 3$, $p = p_o / 2$.

Equation (11) possesses a pulse soliton solution for $y(x, t)$; the well-known form of the Boussinesq soliton solution (here omitted; see in Ref. 11 for details, and also Refs. therein) leads, upon inverting back to the displacement $u(x, t)$, to the kink-shaped solitary wave solution

$$u(x, t) = \pm 2 \left(\frac{6h_0}{q_0} \right)^{\frac{1}{2}} \arctan \left[\frac{1}{p_1} \tanh \left(\frac{x - vt}{L_1} + x_0 \right) \right] \quad (12)$$

where $p_1 = \left[\left(\sqrt{p_0^2 + 6(v^2 - c_0^2)q_0} \pm |p_0| \right) / \left(\sqrt{p_0^2 + 6(v^2 - c_0^2)q_0} \mp |p_0| \right) \right]^{1/2}$.

Here, x_0 and v are real constants, which determine the soliton center and velocity, respectively. The soliton width is expressed by $L_1 = 2\sqrt{h_0/(v^2 - c_0^2)} \equiv 2\sqrt{h_0/c_0}(M^2 - 1)^{-1/2}$ (where $M = v/c_0$ denotes the Mach number). Here $L_1 \gg r_0$ is implicitly assumed, in order for the continuum theory to be valid. The two solutions above correspond to a *rarefactive* and a *compressive* localized excitation, propagating in the dust bi-layer; see Fig. 3a. Equation (10) recovers the nonlinear equation of motion in Ref. 18 for $G'_i(G''_i) = 0$, $i = 1, 2, 3$ (first neighbor interactions, FNI). Incorporating SNI (for $G'_i(G''_i) \neq 0$) results in a slight modification of the soliton profile, as seen in Figure 3(a).

B. Korteweg de Vries (KdV) theory

By defining $\xi = x - c_0 t$, $\tau = p_0 t / 2c_0$, we get from (11)

$$\frac{\partial y}{\partial \tau} + y \frac{\partial y}{\partial \xi} + Ay^2 \frac{\partial y}{\partial \xi} + B \frac{\partial^3 y}{\partial \xi^3} = 0 \quad (13)$$

where $A = 3(\sigma_2 G_3 + 16G'_3)r_0 / 2(\sigma_2 G_2 + 8G'_2)$ and $B = r_0(\sigma_4 G_1 + 16G'_1) / 24(G_2 \sigma_2 + 8G'_2)$. The behavior of the *Extended KdV* (EKdV) Equation (13) depends strongly on the sign of the coefficients A and B , which depends on the ratio of masses, as shown in Figure 4 for $G'_i(G''_i) \neq 0$ (SNI). In specific, the product AB is positive for M_1/M_2 above a certain value (cf. Fig. 4), while for $G'_i(G''_i) = 0$ (FNI), AB is always positive. For $AB < 0$, Eq. (13) has been studied as a limit case of the “compound KdV-Burgers” Equation [21]; the solution thus obtained is qualitatively similar to the positive AB case. Here, we shall assume that $AB > 0$ from the outset.

Eq. (13) is an *Extended KdV* (EKdV) Equation, which bears a pulse soliton solution¹¹ for $y(x, t)$ (here omitted). Upon integrating back to the displacement $u(x, t)$, the kink solitary wave solution

$$u_{\pm}(x, t) = \pm 2 \sqrt{\frac{6h_{\circ}}{q_{\circ}}} \arctan \left[\frac{1}{p_2} \tanh \left(\frac{x - vt}{L_2} + x_2 \right) \right] \quad (14)$$

where $p_2 = \left[\left(\sqrt{p_{\circ}^2 + 12c_{\circ}(v - c_{\circ})q_{\circ}} \pm |p_{\circ}| \right) / \sqrt{p_{\circ}^2 + 12c_{\circ}(v - c_{\circ})q_{\circ}} \mp |p_{\circ}| \right]^{1/2}$.

The parameters x_2 and v determine the center and the velocity of soliton, respectively, while the soliton width depends on $L_2 = 2\sqrt{h_{\circ} / 2c_{\circ}(v - c_{\circ})}$. Note that only supersonic solutions are thus predicted, in contrast to the solution of the GBq Eq. (11) above. Figure 3(b) shows the soliton profile related to Eq. (14), which is only slightly different from the GBq solutions depicted in Fig. 3(a).

IV. OPTICAL NONLINEAR WAVEPACKETS

For $a \neq 1$ (optical wave), upon substituting Eqs. (9b) into Eq. (6b), we obtain

$$\begin{aligned} M_2 \frac{\partial^2}{\partial t^2} [a(u + \sigma_1 r_0 u_x + \sigma_2 \frac{r_0^2}{2} u_{xx} + \dots)] = & -2G_3(a-1)^3 u^3 - 2G_1(a-1)u - r_0 \{ 2G_1(\sigma_1 a - 1)u_x \\ & + 4G_2(a-1)u u_x + 6G_3(a-1)^2 (\sigma_1 a - 1)u^2 u_x \} - r_0^2 \{ G_1(\sigma_2 a - 2)u_{xx} - 4G_1'' a u_{xx} + 4G_2(\sigma_1 a - 1)(u_x)^2 \\ & + 4G_2(a-1)u u_{xx} + 3G_3(a-1)^2 (\sigma_2 a - 2)u^2 u_{xx} + 3G_3(a-1)(2\sigma_1^2 a^2 - 4\sigma_1 a + 4)u (u_x)^2 \} \end{aligned} \quad (15)$$

so keeping terms up to order of r_0^2 , we obtain

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} = & \frac{1}{aM_2} \{ -2G_3(a-1)^3 u^3 - 2G_1(a-1)u - r_0 [2G_1(\sigma_1 a - 1)u_x + 4G_2(a-1)u u_x \\ & + 6G_3(a-1)^2 (\sigma_1 a - 1)u^2 u_x] - r_0^2 [G_1(\sigma_2 a - 2)u_{xx} - 4G_1'' a u_{xx} + 4G_2(\sigma_1 a - 1)(u_x)^2 \\ & + 4G_2(a-1)u u_{xx} + 3G_3(a-1)^2 (\sigma_2 a - 2)u^2 u_{xx} + 3G_3(a-1)(2\sigma_1^2 a^2 - 4\sigma_1 a + 4)u (u_x)^2] \} \\ & - \frac{r_0^2 \sigma_1}{M_1} [2G_1 a (\sigma_1 - 1)u_{xx} + 4G_2 a (a-1)((u_x)^2 + u u_{xx}) + 6G_3 a (a-1)^2 (\sigma_1 - 1)(u^2 u_{xx} + 2u (u_x)^2)] \\ & - \frac{r_0 \sigma_1}{M_1} [2G_3(a-1)^3 (3u_x u^2) + 2G_1(a-1)u_x] - \frac{r_0^2 \sigma_2}{2M_1} [2G_3(a-1)^3 (3u_{xx} u^2 + 6u (u_x)^2) + 2G_1(a-1)u_{xx}] \end{aligned} \quad (16)$$

On the other hand, from Eq. (6a) one obtains:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} = & \frac{1}{M_1} \{ 2G_3 (a-1)^3 u^3 + 2G_1 (a-1)u + r_0 [2G_1 a (\sigma_1 - 1)u_x + 4G_2 a (a-1)u u_x \\ & + 6G_3 a (a-1)^2 (\sigma_1 - 1)u^2 u_x] + r_0^2 [G_1 a (\sigma_2 - 2\sigma_1 + 2)u_{xx} + 4G_1' u_{xx} + 4G_2 a^2 (\sigma_1 - 1)(u_x)^2 \\ & + 4G_2 a (a-1)(\sigma_1 - 1)u u_{xx} + 3G_3 a (a-1)^2 (\sigma_2 - 2\sigma_1 + 2)u^2 u_{xx} + 6G_3 a^2 (a-1)(\sigma_1^2 - 2\sigma_1 + 2)u (u_x)^2] \} \end{aligned} \quad (17)$$

Compatibility among the latter two equations is ensured by the conditions

$$a = -M_1 / M_2, \quad \sigma_1 = 1, \quad \sigma_2 = [2M_2 - 4(G_1' M_1 - G_1' M_2) / G_1] / (M_1 + M_2).$$

We thus obtain the differential equation

$$u_{tt} + Q_4 u_{xx} + Q_1 u + Q_2 u^3 + Q_3 u u_x + Q_5 u^2 u_{xx} + Q_6 u (u_x)^2 = 0 \quad (18)$$

where

$$Q_1 = 2G_1 (M_1 + M_2) / M_1 M_2 = \omega_0^2, \quad (19a)$$

$$Q_2 = 2G_3 (M_1 + M_2)^3 / M_1 M_2^3, \quad (19b)$$

$$Q_3 = -4G_2 r_0 (M_1 + M_2) / M_2^2, \quad (19c)$$

$$Q_4 = -r_0^2 (\sigma_2 a G_1 + 4G_1') / M_1 = c_2^2 \quad (19d)$$

(cf. Section 2A above),

$$Q_5 = -3r_0^2 G_3 a (a-1)^2 \sigma_2 / M_1, \quad (19e)$$

$$Q_6 = -6r_0^2 G_3 a^2 (a-1) / M_1. \quad (19f)$$

It is straightforward to show that all quantities are positive, for Debye interactions (although this may change for dressed electrostatic interactions). Notice that Q_1 , Q_3 , Q_5 and Q_6 bear dimensions of a frequency square ($time^{-2}$), while Q_2 has dimensions of $length^{-2}time^{-2}$ and Q_4 has dimensions of a velocity square ($length^2time^{-2}$). We note the appearance of the characteristic velocity c_2 and the gap frequency ω_0 (cf. definitions above), as physically expected; recall that Eq. (18) essentially originates from the (exact) discrete equations of motion for the optic (inverse dispersive) mode, yet in the

continuum approximation. It is straightforward to show that the definition of Q_4 coincides with c_2^2 as defined in Sec. 2A.

Anticipating modulated envelope solutions, we shall now use a multiple scale technique to study Eq. (18). We take

$$u = \sum_{n=1} \varepsilon^n u_n = \sum_{n=1} \varepsilon^n \sum_{m=0}^n [u_{nm} \exp[im(kx - \omega t)] + u_{nm}^* \exp[-im(kx - \omega t)]] , \quad (20)$$

$$\frac{\partial}{\partial x} = \sum_{n=0}^N \varepsilon^n \frac{\partial}{\partial x_n} , \quad (21)$$

and

$$\frac{\partial}{\partial t} = \sum_{n=0}^N \varepsilon^n \frac{\partial}{\partial t_n} , \quad (22)$$

and proceed by substituting the above series into Eq. (18) and isolating terms arising in the equation of motion at each order ε^n . By solving for u , then substituting in the following order ε^{n+1} and so forth, we obtain the m th harmonic amplitudes u_{nm} in each order, along with a compatibility condition up to any given order.

The equation obtained in order ε^1 is

$$\left(\frac{\partial^2}{\partial t_0^2} + Q_4 \frac{\partial^2}{\partial x_0^2} + Q_1 \right) u_1 = 0 . \quad (23)$$

We may assume that $u_1 = \tilde{u} \exp[i(kx - \omega t)] + c.c. \equiv \exp(i\theta) + c.c.$, where ω and $k = 2\pi/\lambda$ denote the carrier wave frequency and wavenumber (λ is the wavelength), respectively.

The dispersion relation obtained from (23) is of the form

$$\omega^2 = Q_1 - Q_4 k^2 = \omega_0^2 - c_2^2 k^2 , \quad (24)$$

which differs from the dispersion relation in Ref. 18 only due to the appearance of G'_1 (SNI). See that (24) corresponds to the long wavelength limit ($k r_0 \ll 1$) of the discrete relation (8), for the upper mode.

To order ε^2 , we have

$$\left(\frac{\partial^2}{\partial t_0^2} + Q_4 \frac{\partial^2}{\partial x_0^2} + Q_1 \right) u_2 + Q_3 u_1 \frac{\partial u_1}{\partial x_0} = - \left(2 \frac{\partial^2 u_1}{\partial t_0 \partial t_1} + 2 Q_4 \frac{\partial^2 u_1}{\partial x_0 \partial x_1} \right). \quad (25)$$

One has to impose the condition of suppression of secular terms in the right hand side, and so we obtain

$$\frac{\partial \tilde{u}}{\partial t_1} + v_g \frac{\partial \tilde{u}}{\partial x_1} = 0, \quad (26)$$

or $\partial \tilde{u} / \partial \xi = 0$, i.e. the amplitude $\tilde{u} \equiv \tilde{u}(\xi, \tau)$, where

$$\xi = x_1 - v_g t_1 = \varepsilon(x - v_g t), \quad (27a)$$

and

$$\tau = t_2 = \varepsilon^2 t, \quad (27b)$$

implying that the slowly varying wave-front travels at the group velocity $v_g = d\omega / dk = -Q_4 k / \omega$. The solution u_2 obtained at this order consists of a second harmonic contribution

$$u_2 = \frac{i k Q_3 \tilde{u}^2}{4\omega^2 - Q_1 + 4k^2 Q_4} \exp[i2(kx - \omega t)] + \text{c.c.} = \frac{i k Q_3}{3\omega_0^2} \tilde{u}^2 \exp[i2(kx - \omega t)] + \text{c.c.}, \quad (28)$$

where we have used (24) in the last step.

We now proceed to the third order in ε . The condition for annihilation of secular terms now takes the nonlinear Schrodinger equation form (NLSE),

$$i \frac{\partial \tilde{u}}{\partial \tau} + P \frac{\partial^2 \tilde{u}}{\partial \xi^2} + Q |\tilde{u}|^2 \tilde{u} = 0, \quad (29)$$

where the dispersion coefficient P is

$$P = \frac{1}{2} \frac{d^2 \omega}{dk^2} = \frac{-Q_4}{2\omega} \left(1 + \frac{Q_4 k^2}{\omega^2} \right) = -\frac{c_2^2}{2\omega} \left(1 + \frac{c_2^2 k^2}{\omega^2} \right), \quad (30)$$

and the nonlinearity coefficient Q is

$$Q = \frac{-1}{2\omega} \left[3Q_2 + \left(Q_6 - 3Q_5 - \frac{Q_3^2}{4\omega^2 - Q_1 + 4k^2 Q_4} \right) k^2 \right]. \quad (31)$$

We note that the sign of P is negative, while the sign of Q may vary; cf. Fig. 5. However, the former is only true in the continuum limit, and should therefore only be taken for granted in the small k region. Indeed, by evaluating the curvature of the upper mode [and thus $P = \partial^2 \omega / (2\partial k^2)$] from the exact (discrete) dispersion relation (8), one obtains a more perplex dependence of P on the wavenumber k , which may affect the sign of P in a significant way; cf. Fig. 6.

It is known that a modulated wavepacket whose amplitude obeys the NLSE (29) is modulationally unstable for $PQ > 0$, and stable for $PQ < 0$ [Refs. 12, 13]. In the former case ($PQ > 0$), energy localization due to modulational instability may lead to the formation of bright-type solitons (envelope pulses, confining the fast carrier wave, moving at –or near– the group velocity, formally equivalent to bright pulses in nonlinear fiber optics), while in the latter case ($PQ < 0$), modulated wavepackets may propagate in the form of dark/grey envelope solitons (localized voids amidst constant values everywhere else). The exact form of these envelope soliton solutions of (29) can be found in the literature (see e.g. Ref. 22 and references therein) and were thus omitted here.

The analytical form of the coefficients P and Q , as well as their product PQ , as defined in (30, 31) above, is depicted in Figs. 5(a,b,c). Since the sign of PQ determines the stability profile of the wave, we see that instability (along with the occurrence of bright-type wavepackets) is prescribed, since both P and Q are negative (viz. $PQ > 0$). This is true whether second-neighbor interaction are, or are not, taken into account (see Fig. 5), although the former case (SNI) seems to lead to smaller absolute values of P and thus narrower solitons, for a given amplitude (we point out that the ratio P/Q determines the soliton width-amplitude product²²). However, these qualitative results are only valid for small values of the

wave number k , i.e. large wave lengths, since the corresponding expressions were derived in a continuum approximation (so, Fig. 5 was restricted to small k values). To gain some insight into the general (discrete) case, one may retain the expression (31) for Q , yet derive the dispersion coefficient $P = d^2\omega/2dk^2$ from the corresponding discrete dispersion relation (8) (for the upper, optic mode). The result is depicted in Figs. 6(a,b,c), where we see that instability is indeed prescribed in both sides of the 1st Brillouin zone (for small k and for k near $\pi/2r_0$), yet interrupted by a stability range in between, which is reduced if SNI are taken into account; see Fig. 6(c). Figure 7 compares the NLSE dispersion coefficient P as derived from (8) (exact, discrete form) and its continuum approximation [from (30)] in the low wave number region, for both cases where second-neighbor interaction are ($G'_i(G''_i) \neq 0$), or are not ($G'_i(G''_i) = 0$), taken into account. Upper curves in each case (with or without SNI) are belong to exact (discrete) one and in very low wave number $\sin(kr_0) \cong kr_0$, both curves coincide with each other.

V. CONCLUSIONS-DISCUSSION

In this paper, we have investigated the occurrence of nonlinear excitations in a one-dimensional Debye bi-crystal, consisting of two different dust grain species. We have assumed that the two species coexist at the same levitation height, where electric forces balance gravity. We have calculated the polynomial coefficients for the “dressed” potential energy up to fourth order nonlinearity, pointing out the effect of taking or not into account second neighbors interaction (SNI). Two modes were found to exist in the linear limit, namely a lower acoustic mode and an upper optic-like mode; the latter (inverse-dispersive) mode is to be distinguished from the (also inverse-dispersive) transverse mode which is associated with off-plane motion in dust monolayer⁸.

By using a continuum approximation, we have derived and compared an extended KdV and a Boussinesq equation(s) for the acoustic mode. The resulting excitations represent a kink-type

longitudinal displacement propagating in the bi-crystal. Both compressive and rarefactive type excitations were shown to exist.

Considering the upper (optic) mode, we have employed a reductive perturbation technique, in order to derive a nonlinear Schroedinger-type equation for the amplitude of the (quasi-harmonic) longitudinal displacement of dust-grains. An analytical solution was obtained via this perturbative approach, representing a modulated wavepacket, consisting of a slowly varying localized envelope confining the internal carrier wave. The modulated wavepackets were shown to be modulationally unstable in general, so bright-type envelope solitons should be the dominant entity in dynamics. On the other hand, considering a discrete lattice description, a stability range of wavenumber values was shown to exist, for short wavelengths (comparable to the lattice spacing).

We may stress that our results refer to longitudinal motion in 1D dust (bi-)crystals (which have to be created in appropriate experiments making use of trapping potentials^{5,6}). Therefore, this physical situation is distinct from the picture arising when two different dust species “float” at different heights - see, e.g., [Ref. 23] (this case will be treated in forthcoming work). Furthermore, the real dynamics of 2D dust layers includes combined in- and off-plane motion, giving rise, e.g. to the bending modes discussed in Ref. 20 and/or to the transverse (inverse-optic) mode discussed in Ref. 8 (also see in Ref. 25 for details). These effects were not considered here, as we have focused on purely 1D (longitudinal) dust grain motion.

Our investigation has been motivated by recent experiments on dusty plasmas comprising two distinct types of dust grains. Although a co-existence of both species at the same levitation height has admittedly not yet been observed, this is a priori not to be excluded. Our results may thus be of importance in elucidating the dynamics of an anticipated strongly-coupled configuration of such plasma systems.

ACKNOWLEDGMENTS

One of us (BF) acknowledges financial support from Arak-University (Contract No. 11/9017). IK acknowledges support from the FWO (*Fonds Wetenschappelijk Onderzoek-Vlaanderen*, Flemish Research Fund, Belgium) during the course of this work, which was carried out during a short-term Research Associate appointment at the University of Gent, Belgium. Prof. F. Verheest is herewith warmly thanked for actively encouraging this research visit. IK also acknowledges support from the Deutsche Forschungsgemeinschaft (DFG, Germany) under the Emmy-Noether program (grant SH 93/3-1), during the latter stages of this work.

This research was initiated while the authors were attending the *2006 Autumn College on Plasma Physics* at the Abdus Salam ICTP (International Centre of Theoretical Physics), in Trieste. The hospitality and support of the latter are gratefully acknowledged.

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Figure Captions

Figure 1: One-dimensional bi-lattice of dust grains (heuristic representation).

Figure 2: The (square of the) harmonic vibration frequency, i.e., $\omega_{\pm}^2$ (normalized by G_1/μ), is depicted vs the reduced wave number kr_0 . The continuum curve represents SNI and $M_1/M_2 = 5$, the long dashed curve represents SNI and $M_1/M_2 = 3$ and the short-dashed curve represents FNI for $M_1/M_2 = 5$, $G'_i(G''_i) = 0$. The ratio of charges is $Q_1/Q_2 = 2.0$ everywhere.

Figure 3: a) The GBq solitons (dust grain displacement), as defined in Eq. (12), are depicted as a function of position x ($x_0 = t = 0$) for first neighbors (solid curve) and second neighbors (dashed curve). b) The EKdV kink solitons (dust grain displacement), as defined in Eq. (14), are depicted vs position x ($x_0 = t = 0$) for first neighbors (solid curve) and second neighbors (dashed curve). Here, we have assumed Debye interactions with $\kappa = 1$, Mach number $M = v/c_0 = 1.2$; also $M_1 = 2M_2 = 1$, $G = r_0 = 1$ (arbitrary values). Debye length is $\lambda_D = 1$ (i.e., length normalized by λ_D). The ratio of charges is $Q_1/Q_2 = 2.0$.

Figure 4: The coefficient product AB is shown as a function of M_1/M_2 for SNI (solid line) and FNI (dashed line), for two different values of the charge ratio. Here, we have assumed Debye interactions with $\kappa = 1$, $G = r_0 = 1$ (arbitrary values). The Debye length is $\lambda_D = 1$ (i.e., length normalized by λ_D).

Figure 5: (Continuum approximation.) The NLSE coefficients P, Q and PQ, as given by (30,31), as a function of kr_0 , for FNI and SNI. The lattice parameter $\kappa = r_0/\lambda_D$ and the ratio of masses M_1/M_2 are

chosen equal to 1.0 and 2, respectively. Debye length is $\lambda_D = 1$ (i.e., length normalized by λ_D). The ratio of charges is $Q_1 / Q_2 = 2.0$.

Figure 6: (Discrete lattice description.) The NLSE coefficients P [derived from (8)], Q [from (31)] and PQ, as a function of kr_0 for FNI (dashed lines) and SNI (solid lines) relying on Eq. (8). The lattice parameter $\kappa = r_0 / \lambda_D$ and the ratio of masses M_1 / M_2 are chosen equal to 1.0 and 2, respectively. Debye length is $\lambda_D = 1$ (i.e., length normalized by λ_D). The ratio of charges is $Q_1 / Q_2 = 2.0$.

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(a)

(b)

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