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Generalized compound transport of charged particles in turbulent magnetized plasmas

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Abstract

The transport of charged particles in partially turbulent magnetic systems is investigated from first principles. A generalized compound transport model is proposed, providing an explicit relation between the mean-square deviation of the particle parallel and perpendicular to a magnetic mean field, and the mean-square deviation which characterizes the stochastic field-line topology. The model is applied in various cases of study, and the relation to previous models is discussed.

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1. Introduction

The analytical description of magnetic field-line wandering, or *random walk*, in partially turbulent systems is a long standing problem in astrophysics, space physics and turbulence physics. In a number of previous works (e.g., Krommes *et al* (1983), Matthaeus *et al* (1995), Ruffolo *et al* (2004), Ragot (2006), Shalchi and Kourakis (2007)), detailed linear and nonlinear calculations have been performed in order to understand the field-line random walk, in association with the random behavior of dynamical charged particle trajectories in turbulent plasma systems.

The transport of charged particles perpendicular to a large scale magnetic field (e.g., the magnetic field of the Sun, if particle propagation in the solar system is investigated) is often modeled in relevance with field-line wandering. In certain studies, it was assumed that field-line wandering behaves diffusively, yet without giving a justification for this assumption (e.g. Kóta and Jokipii (2000), Webb *et al* (2006)). Other theoretical approaches, such as the nonlinear guiding-center theory (Matthaeus *et al* 2003), the extended NLGC theory (Shalchi 2006) or the weakly nonlinear theory (Shalchi *et al* 2004) have extended that methodology, yet neither using any assumptions about field-line wandering nor assuming that the field-line behavior has a direct influence onto particle propagation.

An open issue in the cosmic ray transport theory is the subdiffusive behavior of perpendicular transport in slab models and the recovery of diffusion for non-slab geometry, as was observed in test-particle simulations (e.g. Qin *et al* 2002a, 2002b). Although the extended nonlinear guiding-center theory (Shalchi 2006) can explain subdiffusion in the slab model as well as the recovery of diffusion in slab/2D composite geometry quantitatively, no physical explanation is yet available, to our knowledge, for these different regimes to be understood. In this paper, we address this problem by relating field-line transport coefficients with particle transport parameters. Via a generalized compound diffusion model, relying on a flexible parametrization of the field-line diffusion and of (in relation with) the actual particle trajectory diffusion in the directions parallel and perpendicular to the magnetic field, we aim to show that different particle random walk regimes obtained in the past come out to be precisely the consequence of underlying assumptions concerning the random behavior of magnetic field lines. The outcome of this study will be important in the theoretical interpretation of charged cosmic ray transport, as provided by space observations.

2. Random walk of magnetic field lines

We shall consider a collisionless magnetized plasma system which is embedded in a uniform mean field ($\vec{B}_0 = B_0\vec{e}_z$) in addition to a turbulent magnetic field component $\delta\vec{B}$. The field-line equation in this system reads $\mathrm{d}x/\mathrm{d}z = \delta B_x/B_0$. Here we assumed a vanishing parallel component of the turbulent field $\delta B_z = 0$. We also assume that the mean field $\vec{B}_0$ is aligned parallel to the z-axis of our (cartesian) system of coordinates.

A characteristic quantity to describe field-line random-walk (FLRW) is the mean-square displacements (MSD's) $\langle(\Delta x)^2\rangle$ and $\langle(\Delta y)^2\rangle$ for large values of z. In the following we only consider the variable x, since all calculations can easily be repeated for y. For axisymmetric turbulence, which is assumed to be a good approximation for real turbulent systems, we have $\langle(\Delta x)^2\rangle = \langle(\Delta y)^2\rangle$. In this case the results derived in this paper for $\langle(\Delta x)^2\rangle$ can also be used for $\langle(\Delta y)^2\rangle$.

By doing this, one anticipates an asymptotic behavior in the following form:

$$\langle(\Delta x)^2\rangle|_{z\to\pm\infty} = \alpha|z|^\beta. \tag{1}$$

Field-line wandering is thus characterized by identifying different parameter regimes for β:

$$\begin{aligned} &0 < \beta < 1: && \text{subdiffusion} \\ &\beta = 1: && \text{normal (Markovian) diffusion} \\ &1 < \beta < 2: && \text{superdiffusion} \\ &\beta = 2: && \text{ballistic behavior.} \end{aligned} \tag{2}$$

In the past several approaches have been proposed to describe FLRW analytically.

In the so-called slab turbulence approach (i.e. assuming that the turbulent fields only depend on the parallel position variable, namely $\delta B_i(\vec{x}) = \delta B_i(z)$, for $i = x, y$) the field-line MSD can be calculated exactly. By assuming a constant wave-spectrum at large turbulence scales (energy range) we find a diffusive behavior of the field lines:

$$\langle(\Delta x)^2\rangle = 2\kappa_{\mathrm{FL}}|z| \tag{3}$$

with the field-line diffusion coefficient κ_{FL}.

The description of FLRW in non-slab turbulence models is more problematic. As an example, we consider the so-called two-component turbulence model, where we have a hybrid combination of the slab and 2D fluctuations (in the latter model, one assumes that $\delta B_i(\vec{x}) = \delta B_i(x, y)$, for $i = x, y$). In the slab/2D composite model we have, precisely,

$$\delta B_i(\vec{x}) = \delta B_i^{(\mathrm{slab})}(z) + \delta B_i^{(2D)}(x, y) \tag{4}$$

(for $i = x, y$). In this case the field-line equation takes the nonlinear form

$$\mathrm{d}x(z) = \frac{\delta B_x^{(\mathrm{slab})}(z)}{B_0}\,\mathrm{d}z + \frac{\delta B_x^{(2D)}(x, y)}{B_0}\,\mathrm{d}z, \tag{5}$$

and an analogous relation holds for the y-component.

In our knowledge, at least three different theories have been developed to describe FLRW analytically:

(i) Quasilinear theory (QLT, Jokipii (1966)) consists in replacing the field-line equation on the right-hand side of equation (5) by the unperturbed lines (i.e. the rectilinear magnetic field topology in the absence of turbulence), say at $x = y = 0$. For pure slab geometry $\left(\delta B_x^{(2D)}(x, y) = 0\right)$, the QLT for FLRW is exact. However, for pure 2D turbulence we have within the QLT

$$\mathrm{d}x(z) = \frac{\delta B_x^{(2D)}(0, 0)}{B_0}\,\mathrm{d}z = \frac{\delta B_x^{(2D)}}{B_0}\,\mathrm{d}z \tag{6}$$

and thus we find for the MSD,

$$\langle(\Delta x)^2\rangle = \frac{\delta B_x^2}{B_0^2} z^2. \tag{7}$$

This result is also obtained for slab/2D composite geometry within the QLT, since the second term in equation (5) is dominant.

(ii) Matthaeus *et al* (1995) proposed a non-perturbative approach based on three *ad hoc* assumptions (namely, the so-called Corrsin hypothesis, Gaussian statistics for the field lines and diffusive FLRW behavior). The following form of the field-line diffusion coefficient

$$\kappa_{\mathrm{FL}} = \frac{\kappa_{\mathrm{FL,slab}} + \sqrt{\kappa_{\mathrm{FL,slab}}^2 + 4\kappa_{\mathrm{FL,2D}}^2}}{2} \tag{8}$$

is thus deduced. Here, $\kappa_{\mathrm{FL,slab}}$ is the pure slab field-line diffusion coefficient and $\kappa_{\mathrm{FL,2D}}$ is the pure 2D field-line diffusion coefficient.

(iii) An improved nonlinear theory for field-line wandering has recently been proposed by Shalchi and Kourakis (2007). In comparison to the Matthaeus *et al* (1995) approach, the authors still rely on the validity of the Corrsin hypothesis and on field-line Gaussian statistics. However, instead of applying a diffusion model as used by Matthaeus *et al* (1995), an ordinary differential equation (ODE) is derived for the field-line MSD. By solving this ODE analytically in the limit $|z| \to \infty$, it is deduced, for slab/2D composite turbulence geometry,

$$\langle(\Delta x)^2\rangle = \left[9C(\nu)\sqrt{\frac{\pi}{2}} l_{2D}\right]^{2/3} |z|^{4/3}, \tag{9}$$

which is clearly a superdiffusive result. Here $C(\nu)$ is a normalization function which depends on the inertial range spectral index 2ν, l_{2D} is the 2D bendover scale of the turbulence (this parameter indicates the turnover from the energy range to the inertial range of the spectrum), and $\delta B_{2D}^2 / B_0^2$ denotes the relative strength of the turbulent magnetic fields. The theory recently presented by Shalchi and Kourakis (2007) is essentially a generalization of the Matthaeus *et al* (1995) approach, yet relies on minimum physical assumptions, thus the description of turbulence by equation (9) may be considered to be more reliable than the diffusive result (8).

In principle we know three different results of FLRW:

- $\langle(\Delta x)^2\rangle \sim z^2$: this result can be found in the initial free-streaming regime and within the QLT for two-component (composite) turbulence.
- $\langle(\Delta x)^2\rangle \sim |z|$: the diffusive result can be derived exactly for slab geometry and a constant spectrum at large turbulence scales. Furthermore, Matthaeus *et al* (1995) have claimed that FLRW also behaves diffusively for slab/2D composite geometry.
- $\langle(\Delta x)^2\rangle \sim |z|^{4/3}$: according to Shalchi and Kourakis (2007), the field lines behave superdiffusively in the two-component model.

In the following, we shall combine these different results of FLRW with the guiding-center approximation and various transport models, in view of a critical comparison among different models.

3. The guiding-center approximation

In several previous papers it has been assumed that charged particles follow magnetic field lines (Jokipii 1966, Kóta and Jokipii 2000, Matthaeus *et al* 2003):

$$\tilde{v}_x = v_z \frac{\delta B_x(\vec{x})}{B_0}, \tag{10}$$

where v_z is the parallel velocity of the charged particle and $\tilde{v}_x$ the perpendicular velocity of its guiding center. This equation can easily be deduced from the field-line equation which reads $\mathrm{d}x = \mathrm{d}z\, \delta B_x/B_0$. A formula which is equivalent to equation (10) is

$$\sigma_\perp(t) = \int_{-\infty}^{+\infty} \mathrm{d}z\, \sigma_{\mathrm{FL}}(z) f_\parallel(z,t). \tag{11}$$

Here we have defined the mean-square deviation (MSD) of the particle displacement in the perpendicular direction

$$\sigma_\perp(t) = \langle(\Delta x(t))^2\rangle, \tag{12}$$

the MSD of the field lines

$$\sigma_{\mathrm{FL}}(z) = \langle(\Delta x(z))^2\rangle_{\mathrm{FL}} \tag{13}$$

and the parallel particle distribution function $f_\parallel(z,t)$. It should be noted that an equivalent formula is given by Krommes *et al* (1983). In the following we discuss two models for $f_\parallel(z,t)$.

4. A Gaussian transport model for parallel scattering

We may assume a Gaussian particle distribution function

$$f_{\parallel,G}(z,t) = \frac{1}{\sqrt{2\pi\sigma_\parallel(t)}}\, \mathrm{e}^{-z^2/(2\sigma_\parallel(t))}, \tag{14}$$

where we have employed the particle MSD in the parallel direction

$$\sigma_\parallel = \langle(\Delta z(t))^2\rangle. \tag{15}$$

Equation (11) thus becomes

$$\sigma_\perp(t) = \frac{1}{\sqrt{2\pi\sigma_\parallel(t)}} \int_{-\infty}^{+\infty} \mathrm{d}z\, \sigma_{\mathrm{FL}}(z)\, \mathrm{e}^{-z^2/(2\sigma_\parallel(t))}. \tag{16}$$

4.1. General results

To proceed with, we may assume the form

$$\sigma_{\mathrm{FL}}(z) = \alpha_{\mathrm{FL}}|z|^{\beta_{\mathrm{FL}}} \tag{17}$$

to obtain

$$\sigma_{\perp}(t) = \frac{\alpha_{\mathrm{FL}}}{\sqrt{2\pi\sigma_{\parallel}(t)}} \int_{-\infty}^{+\infty} \mathrm{d}z|z|^{\beta_{\mathrm{FL}}}\, \mathrm{e}^{-z^2/(2\sigma_{\parallel}(t))}. \tag{18}$$

This integral can easily be solved (Gradshteyn and Ryzhik 2000), so one gets

$$\sigma_{\perp}(t) = \frac{\alpha_{\mathrm{FL}}}{\sqrt{\pi}}\Gamma\left(\frac{\beta_{\mathrm{FL}}+1}{2}\right)(2\sigma_{\parallel}(t))^{\beta_{\mathrm{FL}}/2}, \tag{19}$$

where we used the gamma function $\Gamma(x)$. Furthermore, we assume the forms

$$\sigma_{\parallel}(t) = \alpha_{\parallel}t^{\beta_{\parallel}}, \qquad \sigma_{\perp}(t) = \alpha_{\perp}t^{\beta_{\perp}}. \tag{20}$$

By comparing with equation (19) we obtain

$$\alpha_{\perp} = \frac{\alpha_{\mathrm{FL}}}{\sqrt{\pi}}\Gamma\left(\frac{\beta_{\mathrm{FL}}+1}{2}\right)(2\alpha_{\parallel})^{\beta_{\mathrm{FL}}/2} \tag{21}$$

and, more important by,

$$\beta_{\perp} = \frac{\beta_{\parallel}\beta_{\mathrm{FL}}}{2} \tag{22}$$

for consistency. We see that the independent parameters appearing in (21) and (22) can be used to fine-tune and distinguish different versions of the statistical theory of turbulence, and possibly explain the different results obtained via different assumptions. Our ambition in the following is to pin-point this possibility, by employing specific paradigms. For this purpose, we shall distinguish three different cases, for the field-line statistics, in the following paragraphs.

4.2. Diffusive behavior of FLRW and parallel transport

For pure slab geometry and assuming a constant spectrum in the energy range, it can be shown that field-line wandering behaves diffusively, and thus $\beta_{\mathrm{FL}} = 1$. If we additionally assume that parallel transport also behaves diffusively, namely $\beta_{\parallel} = 1$, we find

$$\alpha_{\perp} = \alpha_{\mathrm{FL}}\sqrt{\frac{2\alpha_{\parallel}}{\pi}} \tag{23}$$

and

$$\beta_{\perp} = \tfrac{1}{2}. \tag{24}$$

For the diffusive field-line behavior we may introduce the field-line diffusion coefficient κ_{FL} via

$$\sigma_{\mathrm{FL}}(z) = 2\kappa_{\mathrm{FL}}|z| \tag{25}$$

and thus $\alpha_{\mathrm{FL}} = 2\kappa_{\mathrm{FL}}$, and the parallel diffusion coefficient $\kappa_{\parallel}$ of the particle position displacement via

$$\sigma_{\parallel}(t) = 2\kappa_{\parallel}t; \tag{26}$$

hence $\alpha_{\parallel} = 2\kappa_{\parallel}$. Therefore we find

$$\alpha_{\perp} = 4\kappa_{\mathrm{FL}}\sqrt{\frac{\kappa_{\parallel}}{\pi}}. \tag{27}$$

Obviously one gets

$$\sigma_\perp(t) = 4\kappa_{\mathrm{FL}}\sqrt{\frac{\kappa_\parallel t}{\pi}} \tag{28}$$

which is clearly a non (classical) diffusive result.

Concluding this paragraph, we have seen that a direct consequence of having assumed diffusive field-line wandering and diffusive particle propagation in the parallel direction is a subdiffusive result for perpendicular particle transport, in the form $\sigma_\perp(t) \sim \sqrt{t}$. This coincides with the result(s) obtained by Krommes *et al* (1983), also within the compound diffusion model of Kóta and Jokipii (2000), and via the extended nonlinear guiding-center theory (Shalchi, 2006).

4.3. Free streaming of field lines

For small length scales the field-line mean-square deviation reads

$$\sigma_{\mathrm{FL}}(z) = \frac{\delta B_x^2}{B_0^2} z^2 \tag{29}$$

and thus

$$\alpha_{\mathrm{FL}} = \frac{\delta B_x^2}{B_0^2} \qquad \text{and} \qquad \beta_{\mathrm{FL}} = 2. \tag{30}$$

Substituting into equations (21) and (22), we find in this case

$$\alpha_\perp = \frac{\delta B_x^2}{B_0^2}\alpha_\parallel, \qquad \beta_\perp = \beta_\parallel. \tag{31}$$

Parallel and perpendicular charged particle transport therefore present the same time behavior if the field lines can be described by equation (29).

4.4. The 4/3-*result of Shalchi and Kourakis (2007)*

According to the results of an improved theory for field-line wandering, recently proposed by Shalchi and Kourakis (2007), we find, for slab/2D turbulence:

$$\beta_{\mathrm{FL}} = 4/3. \tag{32}$$

As a consequence, we find from (22)

$$\frac{\beta_\perp}{\beta_\parallel} = \frac{2}{3}. \tag{33}$$

It has been argued in several previous papers (see Qin *et al* 2002a, 2002b), by using test-particle simulations, that parallel and perpendicular transport behave diffusively in the case of the two-component (composite) model. The question of how this recovery of diffusion can be explained remains unanswered. However, by assuming a symmetric deviation of the diffusive regime of parallel and perpendicular transport

$$\beta_\perp = 1 - \epsilon, \qquad \beta_\parallel = 1 + \epsilon \tag{34}$$

(here we assumed a weak subdiffusive behavior of perpendicular transport and a weak superdiffusive behavior of parallel transport), it can easily be shown that

$$\epsilon = \frac{2 - \beta_{\mathrm{FL}}}{2 + \beta_{\mathrm{FL}}} \tag{35}$$

from (22), and thus for $\beta_{\mathrm{FL}} = 4/3$ this becomes

$$\epsilon = 0.2. \tag{36}$$

This very weak deviation from the diffusive regime cannot be excluded by test-particle simulations.

5. A ballistic transport model for parallel scattering

An alternative to the Gaussian distribution hypothesis adopted in the previous section is a ballistic model of the form:

$$f_{\parallel,B}(z,t) = \delta\,(z - z_0(t)) \tag{37}$$

where we have denoted the unperturbed orbit $z_0(t) = v\mu t$, defining the particle trajectory's pitch-angle cosine μ. In the unperturbed system ($\delta B_x = \delta B_y = 0$) the pitch angle and therefore the parameter μ are conserved. Equation (11) with equation (37) for $f_\parallel(z,t)$ becomes

$$\sigma_\perp(t) = \sigma_{\mathrm{FL}}(z = z_0(t)) \equiv \sigma_{\mathrm{FL}}(z = v\mu t). \tag{38}$$

5.1. General results

Again we assume the form of equation (17) for σ_{FL} to find

$$\sigma_\perp(t) = \alpha_{\mathrm{FL}}(vt|\mu|)^{\beta_{\mathrm{FL}}}. \tag{39}$$

By assuming the form of equation (20) for $\sigma_\perp$ we can deduce

$$\begin{aligned} \alpha_\perp &= \alpha_{\mathrm{FL}}(v|\mu|)^{\beta_{\mathrm{FL}}}, \\ \beta_\perp &= \beta_{\mathrm{FL}}. \end{aligned} \tag{40}$$

Therefore, within the ballistic model, the time exponent of the perpendicular MSD and the length exponent of the field-line MSD are the same.

In the following, we shall consider, two of the three examples exposed above, for the sake of comparison.

5.2. Diffusive behavior of field-line wandering

Applying equations (25) and (37) we get

$$\sigma_\perp = 2\kappa_{\mathrm{FL}} vt|\mu|. \tag{41}$$

Because μ itself is a statistic quantity with $-1 \leqslant \mu \leqslant +1$, the formula can be averaged by integrating with respect to μ, setting

$$\sigma_\perp = 2\kappa_{\mathrm{FL}} vt \left(\frac{1}{2}\int_{-1}^{+1} \mathrm{d}\mu|\mu|\right); \tag{42}$$

hence

$$\sigma_\perp = \kappa_{\mathrm{FL}} vt. \tag{43}$$

Therefore, within the ballistic model and for a diffusive behavior of FLRW we find the well-known quasilinear result for perpendicular transport often referred to as FLRW limit (Jokipii 1966). For the perpendicular diffusion coefficient $\kappa_\perp$ we thus find

$$\kappa_\perp = \frac{v}{2}\kappa_{\mathrm{FL}}, \tag{44}$$

so the perpendicular particle transport coefficient $\kappa_\perp$ is efficiently associated with the field-line diffusion coefficient κ_{FL}.

5.3. Free streaming of field lines

Here, we combine equation (29) with equation (38) to find

$$\sigma_\perp = (v\mu t)^2 \frac{\delta B_x}{B_0} \tag{45}$$

and thus

$$\sigma_\perp = \frac{1}{3}\frac{\delta B_x^2}{B_0^2} v^2 t^2, \tag{46}$$

where the pitch-angle cosine variable μ was again averaged out. Equation (46) corresponds to a ballistic motion of charged particles. Furthermore, this formula is in agreement with the quasilinear result for perpendicular transport obtained by Shalchi and Schlickeiser (2004) for two-component turbulence.

6. General results in the compound transport formulation

In the last two sections, we have discussed two specific models for the parallel particle distribution function, namely the Gaussian model and the ballistic model. In this section, we shall discuss some general properties of the compound transport model, that is, without specifying the form of the parallel distribution function.

6.1. The initial free-streaming regime

A free-streaming-like behavior of field lines is found in some cases (e.g., for small length scales, or if QLT for FLRW is applied for slab/2D composite geometry). In this case we can combine equations (29) and (11) to get

$$\begin{aligned}\sigma_\perp(t) &= \frac{\delta B_x^2}{B_0^2}\int_{-\infty}^{+\infty} \mathrm{d}z\, z^2 f_\parallel(z,t) \\ &= \frac{\delta B_x^2}{B_0^2}\sigma_\parallel(t)\end{aligned} \tag{47}$$

regardless of the form of $f_\parallel(z,t)$. Therefore, for free-streaming of field lines we find

$$\frac{\sigma_\perp(t)}{\sigma_\parallel(t)} = \frac{\delta B_x^2}{B_0^2}. \tag{48}$$

In this case the temporal behavior of parallel and perpendicular transport are the same. In the solar wind at a 1 AU heliocentric distance, we have $\delta B_x^2 \approx B_0^2$. Thus, the mean free path perpendicular to the mean field becomes comparable to the parallel mean free path. For length scales where we have free-streaming of field lines we thus find strong perpendicular scattering of charged cosmic rays.

6.2. Diffusive behavior of FLRW and parallel transport

In section 4.2 we combined the Gaussian model with a diffusive behavior of parallel transport and FLRW to demonstrate that we obtain subdiffusion in the form $\sigma_\perp \sim \sqrt{t}$. In the following, we shall show that the assumption of Gaussian statistics is not necessary to get the subdiffusive perpendicular transport of charged particles. Upon differentiating the basic compound transport relation (11), one gets

$$\frac{\partial}{\partial t}\sigma_\perp(t) = \int_{-\infty}^{+\infty} \mathrm{d}z\, \sigma_{\mathrm{FL}}(z) \frac{\partial f_\parallel(z,t)}{\partial t}. \tag{49}$$

Diffusion in the parallel direction may be assumed, for tractability. Thus, the function $f_\parallel(z,t)$ satisfies the diffusion equation

$$\frac{\partial f_\parallel(z,t)}{\partial t} = \kappa_\parallel \frac{\partial^2 f_\parallel(z,t)}{\partial z^2}. \tag{50}$$

Furthermore, running diffusion coefficients can be introduced by

$$\begin{aligned} d_\perp(t) &\equiv \frac{1}{2}\frac{\partial}{\partial t}\sigma_\perp(t), \\ d_{\mathrm{FL}}(t) &\equiv \frac{1}{2}\frac{\partial}{\partial z}\sigma_{\mathrm{FL}}(z). \end{aligned} \tag{51}$$

Thus equation (49) becomes

$$\begin{aligned} d_\perp(t) &= \kappa_\parallel \int_0^\infty \mathrm{d}z\, \sigma_{\mathrm{FL}}(z) \frac{\partial^2 f_\parallel(z,t)}{\partial z^2} \\ &= -2\kappa_\parallel \int_0^\infty \mathrm{d}z\, d_{\mathrm{FL}}(z) \frac{\partial f_\parallel(z,t)}{\partial z} \\ &= 2\kappa_\parallel \int_0^\infty \mathrm{d}z \frac{\partial d_{\mathrm{FL}}(z)}{\partial z} f_\parallel(z,t), \end{aligned} \tag{52}$$

where we applied $d_{\mathrm{FL}}(z=0)=0$. Now we assume diffusion of field lines, so we have

$$d_{\mathrm{FL}}(z) = \kappa_{\mathrm{FL}} - \epsilon(z) \tag{53}$$

with

$$\epsilon(z=0) = \kappa_{\mathrm{FL}}, \qquad \epsilon(z\to\infty)\to 0. \tag{54}$$

Because of $\partial d_{\mathrm{FL}}(z)/\partial z = -\epsilon'(z)$ (the prime denotes differentiation) we have

$$d_\perp(t) = -2\kappa_\parallel \int_0^\infty \mathrm{d}z\, \epsilon'(z) f_\parallel(z,t). \tag{55}$$

However, if the field lines behave diffusively, the function $\epsilon(z)$ (and therefore also $\epsilon'(z)$) must decay rapidly with increasing z. Thus one gets

$$\begin{aligned} d_\perp(t) &= -2\kappa_\parallel f_\parallel(z=0,t) \int_0^\infty \mathrm{d}z\, \epsilon'(z) \\ &= 2\kappa_\parallel[\epsilon(z=0) - \epsilon(z=\infty)] f_\parallel(z=0,t). \end{aligned} \tag{56}$$

Combining with equation (54), we deduce

$$d_\perp(t) = 2\kappa_\parallel \kappa_{\mathrm{FL}} f_\parallel(z=0,t). \tag{57}$$

As an example, we may consider again the Gaussian transport model. Evaluating equation (14) at $z=0$, one can easily recover equation (28) as a special limit of equation (57). However, equation (57) is more general, and can also be applied on non-Gaussian statistics. In real physical systems, one expects that the probability to find the particle at $z=0$ decreases with time ($f_\parallel(z=0, t\to\infty)\to 0$), so consequently

$$d_\perp(t) \to 0, \tag{58}$$

which is interpreted as subdiffusion. Thus, for parallel diffusion of charged particles in combination with a diffusive behavior of FLRW, we find subdiffusion in the perpendicular direction.

7. Summary and conclusion

In this paper we have discussed the generalized compound diffusion mechanism, which relates field-line statistics (wandering) to charged particle random walk in real (position) space. By applying the guiding-center approximation (equation (11)) and a Gaussian transport model for parallel scattering (equation (14)) we deduced the general relation

$$\beta_\perp = \frac{\beta_\parallel \beta_{\mathrm{FL}}}{2}. \tag{59}$$

Here $\beta_\perp$ and $\beta_\parallel$ are the time exponents of the perpendicular and parallel MSD's of the particles whereas β_{FL} is the length exponent of the field-line MSD.

For diffusive FLRW ($\beta_{\mathrm{FL}} = 1$) and diffusive parallel motion of charged particles ($\beta_\parallel = 1$), we always find subdiffusion in the form $\sigma_\perp(t) \sim \sqrt{t}$. This relation was derived in several previous papers (e.g. Krommes *et al* (1983), Kóta and Jokipii (2000), Shalchi (2006)). However, as demonstrated in this paper, we always get this subdiffusive behavior if FLRW and parallel transport behave diffusively. Perpendicular diffusion and parallel diffusion can only be obtained for $\beta_{\mathrm{FL}} = 2$—see equation (59)—which corresponds to free-streaming of field lines.

By replacing the Gaussian transport model by a ballistic model, and by assuming diffusive behavior of FLRW one can easily recover the well-known quasilinear result often referred to as FLRW limit (Jokipii 1966). By combining the quasilinear FLRW result for two-component turbulence—see equation (7)—with the ballistic model we find superdiffusion of charged particles in the perpendicular direction. Thus, the generalized compound transport model discussed in this paper is able to obtain the well-known QLT results in appropriate special limits.

By applying the relation $\sigma_{\mathrm{FL}}(z) \sim |z|^{4/3}$, i.e. the superdiffusive result obtained by Shalchi and Kourakis (2007), and by assuming a weakly superdiffusive behavior of parallel transport (e.g. $\beta_\parallel = 1.2$), we find that perpendicular diffusion is nearly recovered (e.g. $\beta_\perp = 0.8$). This recovery of perpendicular diffusion is an effect which can be found in test particle simulations (e.g. Giacalone and Jokipii (1999), Qin *et al* (2002b)). For diffusive behavior of field lines and parallel diffusion of charged particles these simulations cannot be reproduced theoretically. However, the combination of the Shalchi and Kourakis (2007) theory for field-line wandering with the generalized compound transport model is able to describe this effect.

Our results are important in the theoretical interpretation of cosmic ray transport, following (and interpreting) measurements provided by space observations. It will be the subject of future work to compare these new results with test-particle simulations and solar wind observations.

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