Dust lattice wave dispersion relations in two-dimensional hexagonal crystals including the effect of dust charge polarization

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Abstract

A dusty plasma crystalline configuration with equal charge dust grains and mass is considered. Both charge and mass of each dust species are taken to be constant. Two differential equations for a two-dimensional hexagonal crystal on the basis of a Yukawa-type potential energy and a “dressed” potential energy, accounting for dust charge polarization, are derived and compared. The dispersion relation for both longitudinal and transverse wave propagation in an arbitrary direction is derived. A comparison to analytical and experimental results reported previously is carried out.

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Dusty plasma crystals represent strongly coupled dust configurations, typically occurring in plasma discharge experiments, due to the strong electrostatic interaction between massive, heavily charged, micron-sized dust particulates (dust grains) injected into the plasma [1, 2]. Such dust crystals, which most often bear a two-dimensional (2D) hexagonal structure [2], support a variety of linear modes [1-5]. A theoretical treatment of longitudinal and transverse modes in Yukawa crystals including the effects of damping are investigated by X. Wang et al. [6]. Their theoretical predictions are in agreement with experiments. A theoretical analysis, supported by molecular dynamics simulation, of the wave dispersion relation in a 2D dust crystal in the presence of a constant magnetic field, was presented by G. Uchida et al. [7]. Crystal formation and dynamics have been studied in various experiments [2, 8-14], where particles were essentially created by injecting artificial micro-spheres, which subsequently acquire a high (negative, usually) electron charge via inherent dynamic charging mechanisms.

Recently, W. S. Duan et al. [15] have investigated longitudinal and transverse dust grain vibrations in a 2D hexagonal lattice by considering screened Coulomb interactions (Debye-Hueckel or Yukawa system) between charged dust particles, i.e. $U(r) = Q^2 \exp(-r/\lambda_D)/4\pi\varepsilon_0 r$, where $r$ is the distance between two dust particles and $\lambda_D$ is the Debye radius ($\varepsilon_0$ denotes the electric susceptibility of vacuum). Taking into account polarization due to the sheath region (near the grain surface) leads to a strong modification of the charge cloud (of opposite electric charge sign) surrounding the particles. Such a “dressing” effect in the particle interactions may even result in an attractive force between equal-sign charged dust particles [16-18].

In this Letter, we have investigated the dynamics of a two-dimensional hexagonal crystal, in search of a dispersion relation for longitudinal and transverse dust lattice waves (DLWs). The principal aspects of harmonic (linear) small-amplitude vibrational motion are investigated, by considering a “dressed” Debye-type interaction potential energy, namely $U(r) = Q^2 (1 - r/2\lambda_D) \exp(-r/\lambda_D)/4\pi\varepsilon_0 r$.

Let us consider a two-dimensional (2D) hexagonal crystal (assumed infinite, for simplicity) consisting of negative dust grains (of constant charge $Q$ and mass $M$, for simplicity), located at equidistant sites $a$. For the analysis of the waves in this 2D crystalline monolayer, we use the so-called "particle string" model, allowing for two-dimensional motion, in the longitudinal (horizontal,
along the $x$ axis) and transverse (vertical) directions, the corresponding discrete ordering dust grain being denoted by indices $n$ and $m$, respectively. Figure 1 shows the nearest six particles with labels $(n+1,m)$, $(n-1,m)$, $(n+1/2,m+\sqrt{3}/2)$, $(n-1/2,m+\sqrt{3}/2)$, $(n+1/2,m-\sqrt{3}/2)$, and $(n-1/2,m-\sqrt{3}/2)$. Let the $(n,m)$th particle location define the origin of the plane; then the positions of the first elementary cell particles at equilibrium are $(a,0)$, $(-a,0)$, $(a/2,\sqrt{3}a/2)$, $(-a/2,\sqrt{3}a/2)$, $(a/2,-\sqrt{3}a/2)$ and $(-a/2,-\sqrt{3}a/2)$. However, if the particles are not at their equilibrium positions, we then define the six lengths $l_1$, $l_2$, $l_3$, $l_4$, $l_5$, and $l_6$ to represent the distances from particle $(n,m)$ to the nearest particles, respectively,

$$l_1 = \sqrt{(a+u_{n+1,m}-u_{n,m})^2 + (v_{n+1,m}-v_{n,m})^2},$$

$$l_2 = \sqrt{(a+u_{n,m}-u_{n-1,m})^2 + (v_{n,m}-v_{n-1,m})^2},$$

$$l_3 = \sqrt{(a/2+u_{n+1/2,m+\sqrt{3}/2}-u_{n,m})^2 + (\sqrt{3}a/2+v_{n+1/2,m+\sqrt{3}/2}-v_{n,m})^2},$$

$$l_4 = \sqrt{(a/2-u_{n-1/2,m+\sqrt{3}/2}+u_{n,m})^2 + (\sqrt{3}a/2+v_{n-1/2,m+\sqrt{3}/2}-v_{n,m})^2},$$

$$l_5 = \sqrt{(a/2+u_{n+1/2,m-\sqrt{3}/2}-u_{n,m})^2 + (\sqrt{3}a/2-v_{n+1/2,m-\sqrt{3}/2}+v_{n,m})^2},$$

$$l_6 = \sqrt{(a/2-u_{n-1/2,m-\sqrt{3}/2}+u_{n,m})^2 + (\sqrt{3}a/2-v_{n-1/2,m-\sqrt{3}/2}+v_{n,m})^2},$$

where $u$ and $v$ are the particle displacements from their equilibrium positions in the $x$ and $y$ directions, respectively. The electrostatic binary interaction force $F(r)$ exerted onto two dust grains situated at a distance $r$ is derived from a potential function $U(r)$, viz. $F(r) = -\partial U(r)/\partial r$. We may expand the potential energy around equilibrium at $r=a$, viz.

$$U(r)=U(a)+ (r-a) \frac{\partial U}{\partial r}|_{r=a} + \frac{1}{2} (r-a)^2 \frac{\partial^2 U}{\partial r^2}|_{r=a} + \ldots,$$

By defining the “spring” constant $G=(\partial^2 U/\partial r^2)|_{r=a}$, and setting the potential energy at equilibrium to zero, we have

$$U(r) = \frac{1}{2} G (r-a)^2.$$

We have calculated $G$, for a Yukawa system $U_1(r)$ on one hand, and for a “dressed” potential energy $U_2(r)$, on the other; the corresponding expressions read

$$G_1 = \frac{2Q^2}{4\pi \varepsilon_0 \lambda_D^3} \frac{1+\kappa+\kappa^2/2}{\kappa^3} e^{-\kappa},$$

and
\[ G_2 = \frac{2Q^2}{4\pi e_0^2 D^3} \frac{1+\kappa^2/2-\kappa^3/4}{\kappa^3} e^{-\kappa}, \]  

respectively, where we have defined the (dimensionless) lattice parameter \( \kappa = a/\lambda_D \). Figure 2 shows the harmonic potential energy, given by Eq. (8), as a function of \( \delta r/\lambda_D \), where \( \delta r \) denotes the dust grain deviation from equilibrium. Both types of potential energy function \( U(r) \) present a minimum, so particles vibrate around equilibrium. The respective characteristic vibration constants \( G_1 \) and \( G_2 \) have been as a function of \( \kappa \) in Fig. 3. For large enough \( \kappa \), both spring constants tend to zero, while for small \( \kappa \), we have \( G_1 > G_2 \) (notice that \( \kappa = a/\lambda_D > 1 \) here). We note that the second derivative of the dressed potential (viz. \( G_2 \)) changes sign at \( \kappa = 3.48 \); therefore, for \( \kappa \geq 3.48 \), \( \omega^2 \) becomes negative, and so \( \omega \) becomes complex. In this case, the crystal becomes unstable, and melting occurs.

The components of the force exerted on the particle \((n,m)\) by the nearest particles, in the \(x\) and \(y\) directions, are given by

\[
F_x = G \left[ \frac{l_1-a}{l_1} \left( a+u_{n+1,m} - u_{n,m} \right) - \frac{l_2-a}{l_2} \left( a+u_{n,m} - u_{n-1,m} \right) \right] \\
+ G \left[ \frac{l_3-a}{l_3} \left( \frac{a/2+u_{n+1,2,m+\sqrt{3}/2}-u_{n,m}}{l_4-a} \right) \frac{a/2+u_{n,m} - u_{n-1,2,m+\sqrt{3}/2}}{l_4} \right] \\
+ G \left[ \frac{l_5-a}{l_5} \left( \frac{a/2+u_{n+1,2,m-\sqrt{3}/2}-u_{n,m}}{l_6-a} \right) \frac{a/2+u_{n,m} - u_{n-1,2,m-\sqrt{3}/2}}{l_6} \right],
\]

and

\[
F_y = G \left[ \frac{l_1-a}{l_1} \left( v_{n+1,m} - v_{n,m} \right) - \frac{l_2-a}{l_2} \left( v_{n,m} - v_{n-1,m} \right) \right] \\
+ G \left[ \frac{l_3-a}{l_3} \left( \frac{\sqrt{3}a/2+v_{n+1,2,m+\sqrt{3}/2}-v_{n,m}}{l_4-a} \right) \frac{\sqrt{3}a/2+u_{n,m} - u_{n-1,2,m+\sqrt{3}/2}}{l_4} \right] \\
+ G \left[ \frac{l_5-a}{l_5} \left( \frac{\sqrt{3}a/2+u_{n+1,2,m-\sqrt{3}/2}-u_{n,m}}{l_6-a} \right) \frac{\sqrt{3}a/2+u_{n,m} - u_{n-1,2,m-\sqrt{3}/2}}{l_6} \right].
\]

For small amplitude waves, i.e. \( \Delta u_j \ll a \) and \( \Delta v_j \ll a \), we may neglect the nonlinear terms, and thus obtain equations of motion in the \(x\) and \(y\) directions. We have
\[
M \frac{\partial^2 u_{nm}}{\partial t^2} = G \left[ u_{n+1,m} + u_{n-1,m} - 2u_{nm} + \frac{1}{4} \left( u_{n+1/2,m+\sqrt{3}/2} + u_{n-1/2,m+\sqrt{3}/2} ight)
\right]
\]

\[
+ u_{n+1/2,m-\sqrt{3}/2} + u_{n-1/2,m-\sqrt{3}/2} - 4u_{nm} \right] + \frac{\sqrt{3}}{4} \left( v_{n+1/2,m+\sqrt{3}/2} - v_{n-1/2,m+\sqrt{3}/2}
\right)
\]
\[
- v_{n+1/2,m-\sqrt{3}/2} + v_{n-1/2,m-\sqrt{3}/2} \right] \] \label{eq:13}

and
\[
M \frac{\partial^2 v_{nm}}{\partial t^2} = G \left[ \frac{3}{4} \left( v_{n+1/2,m+\sqrt{3}/2} - v_{n-1/2,m+\sqrt{3}/2} + v_{n+1/2,m-\sqrt{3}/2} + v_{n-1/2,m-\sqrt{3}/2} - 4v_{nm} \right)
\right]
\]
\[
+ \frac{\sqrt{3}}{4} \left( u_{n+1/2,m+\sqrt{3}/2} - u_{n-1/2,m+\sqrt{3}/2} - u_{n+1/2,m-\sqrt{3}/2} + u_{n-1/2,m-\sqrt{3}/2} \right) \] \label{eq:14}

In a hexagonal 2D crystal, longitudinal (for \( \nu_{nm} = 0 \)) and transverse (for \( \nu_{nm} = 0 \)) waves can propagate along an arbitrary direction, denoted by an angle \( \theta \), which represents the angle between the wavevector \( k \) and a primitive translation vector (along the \( x \) axis), i.e. \( k_x = k \cos \theta \) and \( k_y = k \sin \theta \). Two independent principal directions are thus defined, namely the parallel \( (\theta = 0) \) and perpendicular \( (\theta = \pi/2) \) to the primitive translation vector of the crystal.

For longitudinal waves and \( \theta = 0 \), letting \( u_{nm} = u_0 \exp \left[ i \left( k_na - \omega t \right) \right] \) and \( \nu_{nm} = 0 \), we obtain the following dispersion relation
\[
\omega_L^2 (\theta = 0) = 2G\omega_0^2 \sin^2 \left( k a / 4 \right) \left[ 1 + 4 \cos^2 \left( k a / 4 \right) \right] \] \label{eq:15}

where we have defined the characteristic frequency \( \omega_0 = (Q^2/4 \pi \varepsilon_0 \kappa_D^3 M)^{1/2} \).

A similar dispersion relation can be obtained for longitudinal waves and \( \theta = \pi/2 \), by using \( \nu_{nm} = \nu_0 \exp \left[ i \left( kma - \omega t \right) \right] \) and \( u_{nm} = 0 \), namely
\[
\omega_L^2 (\theta = \pi/2) = 6G\omega_0^2 \sin^2 (\sqrt{3} ka / 4) \] \label{eq:16}

For transverse waves and \( \theta = 0 \), letting \( u_{nm} = u_0 \exp \left[ i \left( kma - \omega t \right) \right] \) and \( \nu_{nm} = 0 \), we obtain the dispersion relation
\[
\omega_T^2 (\theta = 0) = 2G\omega_0^2 \sin^2 (\sqrt{3} ka / 4) \] \label{eq:17}
and for $\theta=\pi/2$, and letting $v_{nm} = v_0 \exp[i(kn\alpha-\omega t)]$, $u_{nm} = 0$, then

$$\omega^2_{L}(\theta=\pi/2) = 6G\omega_0^2 \sin^2(ka/4).$$

The longitudinal DLW dispersion relations obtained above are depicted in Figs. 4-7, for both plain (non-polarized) and dressed (polarized) interaction potentials; a high lattice constant $\kappa$ value of 2.5 was chosen, in order to emphasize the quantitative effect of polarization. Figures 4 and 5 exhibit that the frequency in the latter case is smaller than the first (non-polarized) one. In the limit $ka \to 0$, the longitudinal dispersion relations are

$$\omega_L(\theta=0) \approx \left[\sqrt{\frac{10G}{4}} \omega_0 a\right]k \equiv C_s k,$$

and

$$\omega_L(\theta=\pi/2) \approx \left[\frac{3\sqrt{2G}}{4} \omega_0 a\right]k \equiv C'_s k,$$

respectively, where $C_s$ and $C'_s$ denote the sound speed, for horizontal and vertical propagation, respectively, in the hexagonal 2D dusty plasma monolayer. We note that the sound speed for longitudinal waves with $\theta=\pi/2$ is smaller than in the $\theta=0$ direction. Since $G_2 < G_1$, the sound speed for dressed potential interactions is smaller than the one obtained for a Yukawa system, for both values of $\theta$ (directions). Figure 6 shows the normalized sound speed $C_s/\omega_0 a$ as a function of $\kappa$ for longitudinal waves.

Considering transverse DLWs, we find that the effect of electric potential dressing results in a net reduction of the oscillation frequency, as seen in Figs. 7 and 8.

These results are in agreement with analytical and experimental results reported previously [13-15]. S. Nunomura et al. [13] have reported the dispersion relation of longitudinal and transverse in 2D screened-Coulomb crystal experimentally. Data on Figs. 6 and 7 of Ref. [13] are in agreement with Figs. 4-7 here, as regards Yukawa crystals. The sound speed and the dispersion relation of waves in a 2D hexagonal Yukawa crystal were obtained by Y. Liu et al. [14], via molecular dynamics simulation, and independently by W. S. Duan et al. [15] theoretically. Figures 3, 5, and 6 of Ref. [14] and Figures 2 and 3 of Ref. [15] coincide with our results, for Yukawa-type interactions.

To summarize, we have analyzed the properties of linear DLWs in the hexagonal two-dimensional lattice on the basis of a “dressed” (charge polarized) potential energy, and we have compared these results with those corresponding to an ordinary Yukawa potential. We have found
that the spring constant for a dressed potential is smaller than for the Yukawa potential. Comparing the frequency dependence on the wavenumber \( k \) (the dispersion relation) for longitudinal and transverse vibrations, propagating either parallel or vertically with respect to the primary crystal translation vector, we see that the result in all cases is a net reduction of the frequency, for all values of the wavenumber \( k \). The sound speed for longitudinal waves with \( \theta = \pi / 2 \) is smaller than in the \( \theta = 0 \) direction. Since \( G_2 < G_1 \), the sound speed for dressed potential interactions is smaller than the one obtained for a Yukawa system, for both values of \( \theta \) (directions).

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References

FIGURE CAPTIONS

Figure 1:
The nearest neighbors around the particle $m, n$ in a particle hexagonal lattice.

Figure 2:
The normalized potential energy $U(r)/(Q^2/4\pi\varepsilon_0\lambda_D^3)$ (here magnified by a factor of one hundred), as given by Eq. (8), is depicted for a Yukawa-type interaction potential, and for a dressed potential energy. The ratio of the equilibrium distance between the adjacent dust particles to the Debye radius $\kappa=a/\lambda_D$ is taken 2.5 for both curves.

Figure 3:
(a) The spring constant $G_1$ for the Yukawa potential, and $G_2$ for the dressed potential energy, as expressed by Eqs. (9) and (10), respectively, are depicted as a function of $\kappa=a/\lambda_D$.
(b) Detail of (a): Note that $G_2$ (viz. the second derivative of the dressed potential) changes sign at $\kappa=3.48$.

Figure 4:
The normalized longitudinal frequency $\omega_L/\omega_0$ is depicted as a function of the normalized wavenumber $ka$, for wave propagation in the $y$ direction ($\theta=\pi/2$). The ratio of the inter-particle distance at equilibrium to the Debye radius $\kappa=a/\lambda_D$ here is 2.5 for both curves: (1) for the Yukawa-type interaction; (2) for a dressed potential energy.

Figure 5:
The normalized longitudinal frequency $\omega_L/\omega_0$ is depicted as a function of the normalized wavenumber $ka$, for wave propagation in the $x$ direction ($\theta=0$). The ratio of the inter-particle distance at equilibrium to the Debye radius here is 2.5 for both curves: (1) for the Yukawa-type interaction; (2) for a dressed potential energy.
Figure 6:

The sound speed (normalized) of longitudinal waves is depicted as a function of $\kappa = a/\lambda_D$ for the Yukawa-type interaction: (a) $\theta = 0$ and (b) $\theta = \pi/2$; for the dressed potential interaction: (c) $\theta = 0$ and (d) $\theta = \pi/2$.

Figure 7:

The normalized transverse wave frequency $\omega_T/\omega_0$ is depicted as a function of the normalized wavenumber $ka$, for wave propagation along the $y$ direction ($\theta = \pi/2$). The ratio of the equilibrium lattice spacing to the Debye radius $\kappa = a/\lambda_D$ is 2.5, for both curves: (1) for the Yukawa-type interaction; (2) for the dressed potential interaction.

Figure 8:

The normalized transverse wave frequency $\omega_T/\omega_0$ is depicted as a function of the normalized wavenumber $ka$, for wave propagation along the $x$ direction ($\theta = 0$). The ratio of the equilibrium lattice spacing to the Debye radius $\kappa = a/\lambda_D$ is 2.5, for both curves: (1) for the Yukawa-type interaction; (2) for the dressed potential interaction.
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