Nonlinearly coupled whistlers and dust-acoustic perturbations in dusty plasmas *

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(Dated: Submitted 19 October 2005; revised 2 November 2005)

Abstract

The nonlinear interaction between magnetic field-aligned coherent whistlers and dust-acoustic perturbations (DAPs) in a magnetized dusty plasma is considered. The interaction is governed by a pair of equations consisting of a nonlinear Schrödinger equation for the modulated whistler wave packet, and an equation for the non-resonant DAPs in the presence of the ponderomotive force generated by the whistlers. The coupled equations are employed to investigate the occurrence of modulational instability, in addition to the formation of whistler envelope solitons. This investigation is relevant to amplitude modulated electron whistlers in magnetized space dusty plasmas.

* Submitted to *Physics of Plasmas*.

In a pair of classic papers, Hasegawa [1, 2] presented a theory for the nonlinear interaction between magnetic field-aligned electromagnetic electron-cyclotron waves (ECWs) and low-frequency magnetohydrodynamic (MHD) perturbations in an electron-ion plasma. He demonstrated the possibility of the modulational instability of large amplitudes ECWs, and presented a criterion for wave localization/breaking. Karpman and Washimi [3] presented a complete theory for the modulational and filamentational instabilities of obliquely propagating ECWs/electron whistlers, taking into account their interaction with fast and slow magnetosonic perturbations in plasmas. Shukla and Stenflo considered the nonlinear coupling between ECWs/electron whistlers and electrostatic ion-sound [4] and shear Alfvén wave perturbations [5].

In this Brief Communication, we consider the nonlinear coupling between right-hand circularly polarized electron whistlers and dust acoustic perturbations (DAPs) [6] in an electron-ion-dust plasma. The electron whistlers, as well as the DAPs, are assumed to propagate along the external magnetic field $B_0\hat{z}$, where B_0 is the strength of the magnetic field and $\hat{z}$ is the unit vector along the z axis.

The constituents of our dusty plasmas are electrons, singly ionized ions, and charged dust grains. At equilibrium we have $n_{i0} = n_{e0} - \epsilon Z_d n_{d0}$, where n_{j0} denotes the equilibrium number density of the particle species j (j equals e for electrons, i for ions, and d for charged dust grains), $\epsilon = -1(+1)$ for negatively (positively) charged dust grains, and Z_d is the number of charges residing on a dust grain. The electric field of the electron whistlers is $\mathbf{E} = E_k(\hat{x} + i\hat{y})\exp(-i\omega_k t + ikz) +$ c.c., where $\hat{x}$ ($\hat{y}$) is the unit vector along the x (y) axis and c.c. stands for the complex conjugate. Assuming $kc \ll \omega_{pe}$ and $\omega_{pi} \ll \omega_k \ll \omega_{ce}$, the wave frequency ω_k is related to the wave number k by

$$\omega_k = \frac{k^2 c^2 \omega_{ce}}{\omega_{pe}^2}, \tag{1}$$

where c is the speed of light in vacuum, $\omega_{ce} = eB/m_e c$ is the electron gyrofrequency, and $\omega_{pe(i)} = (4\pi n_{e(i)} e^2/m_{e(i)})^{1/2}$ is the electron (ion) plasma frequency.

The nonlinear interactions between large-amplitude coherent electron whistlers and DAPs gives rise to modulated envelope waves whose electric field amplitude evolves according to the equation [3, 4]

$$i\left(\frac{\partial}{\partial\tau}+V_g\frac{\partial}{\partial z}\right)E_k+\frac{V_g'}{2}\frac{\partial^2 E_k}{\partial z^2}+\frac{kV_g}{2}\left(N_e-2\frac{V_z}{V_g}\right)E_k=0, \tag{2}$$

where τ and z are slow time and space variables; we have assumed that $\partial E_k/\partial\tau \ll \omega_k E_k$. Here, $V_g=\partial\omega_k/\partial k=2kc^2\omega_{ce}/\omega_{pe}^2$ is the group velocity, $V_g'=2c^2\omega_{ce}/\omega_{pe}^2$ is the group velocity dispersion, $N_e=n_{e1}/n_{e0}$ is the relative electron density perturbation associated with the DAPs, V_z is the free electron streaming velocity of the plasma slow motion, determined from the electron continuity equation

$$\frac{\partial N_e}{\partial\tau}+\frac{\partial V_z}{\partial z}=0. \tag{3}$$

Since the phase speed of the DAPs is much smaller than the electron and ion thermal speeds, we have from the inertialess electron and ion momentum equations

$$0 = e\frac{\partial\varphi}{\partial z}-T_e\frac{\partial N_e}{\partial z}+F_p\,, \tag{4}$$

$$0 = -e\frac{\partial\varphi}{\partial z}-T_i\frac{\partial N_i}{\partial z}\,, \tag{5}$$

where e is the magnitude of the electron charge, φ is the electric potential associated with the DAPs, $N_i=n_{i1}/n_{i0}$ is the relative ion density perturbation associated with the DAPs, T_e (T_i) is the electron (ion) temperatures, and the electron whistler ponderomotive force is [3]

$$F_p=\frac{\omega_{pe}^2}{4\pi n_{e0}\omega\omega_{ce}}\left(\frac{\partial}{\partial z}+\frac{2}{V_g}\frac{\partial}{\partial\tau}\right)|E_k|^2. \tag{6}$$

The whistler ponderomotive force acting on the ion and dust fluids is small. It is transmitted to the ions and the dust grains through the space charge electric field $-\partial\phi/\partial z$. The dust dynamics is governed by the dust continuity and momentum equations

$$\frac{\partial N_d}{\partial\tau}+\frac{\partial(N_dV_d)}{\partial z}=0 \tag{7}$$

$$\frac{\partial V_d}{\partial\tau}=-\epsilon\frac{Z_de}{m_d}\frac{\partial\varphi}{\partial z}-3V_{Td}^2\frac{\partial N_d}{\partial z}, \tag{8}$$

where $N_d=n_{d1}/n_{d0}$ is the relative dust number density perturbation, V_d is the dust fluid velocity, m_d is the dust mass, and $V_{Td}=(T_d/m_d)^{1/2}$ is the dust thermal speed. The system of equations is closed by means of the quasi-neutrality condition

$$n_{e1} - \epsilon Z_d n_{d1} = n_{i1}, \tag{9}$$

which holds for long wavelength [in comparison with the dusty plasma Debye radius $\lambda_D = \lambda_{De}\lambda_{Di}/(\lambda_{Di}^2 + \lambda_{De}^2)^{1/2}$] DAPs, where $\lambda_{Di} = (T_i/4\pi n_{i0}e^2)^{1/2}$ and $\lambda_{De} = (T_e/4\pi n_{e0}e^2)^{1/2}$ are the electron and ion Debye radii. Note the influence of the dust presence, manifested via the ratio $n_{i0}/n_{e0} = 1 - \epsilon Z_d n_{d0}/n_{e0}$ (densities at equilibrium), which may take quite high (low) values, in the presence of negative (positive) dust.

Combining Eqs. (5), (7), (8), using Eq. (9), and assuming $\partial^2 n_{e1}/\partial\tau^2 \gg V_{Td}^2 \partial^2 n_{e1}/\partial z^2$, we have

$$\frac{\partial^2 N_e}{\partial\tau^2} + \frac{n_{i0}e}{n_{e0}T_i}\Big(\frac{\partial^2}{\partial\tau^2} - C_{Di}^2\frac{\partial^2}{\partial z^2}\Big)\varphi = 0\,, \tag{10}$$

where we have denoted $C_{Di} = Z_d\,(n_{d0}T_i/n_{i0}m_d)^{1/2}$.

By using Eq. (4) we can now eliminate φ from Eq. (10), obtaining

$$\Big(\frac{\partial^2}{\partial\tau^2} - C_{DH}^2\frac{\partial^2}{\partial z^2}\Big)\frac{\partial N_e}{\partial z} = \frac{\mu}{I_{th}}\Big(\frac{\partial^2}{\partial\tau^2} - C_{Di}^2\frac{\partial^2}{\partial z^2}\Big)\Big(\frac{\partial}{\partial z} + \frac{2}{V_g}\frac{\partial}{\partial\tau}\Big)|E_k|^2\,, \tag{11}$$

where $C_{DH} = [\sigma C_{Di}^2/(1+\sigma)]^{1/2}$ is the dust acoustic speed, $\sigma = n_{i0}T_e/n_{e0}T_i$, $\mu = \sigma\omega_{pe}^2/(1+\sigma)\omega\omega_{ce}$, and $I_{th} = 4\pi n_{e0}T_e$.

We now proceed by seeking solitary wave structures moving at the group velocity V_g. Employing Eq. (3) in order to eliminating V_z in Eqs. (2) and (11), the latter two equations lead to

$$i\frac{\partial E_k}{\partial\tau} + \frac{V_g'}{2}\frac{\partial^2 E_k}{\partial\zeta^2} - \frac{kV_g}{2}N_e E_k = 0, \tag{12}$$

and

$$N_e = -\frac{\mu}{I_{th}}\,\frac{V_g^2 - C_{Di}^2}{V_g^2 - C_{DH}^2}(|E_k|^2 - |E_\infty|^2)\,, \tag{13}$$

where we have integrated, assuming $E_k \to E_\infty =$ constant, at infinity. In the following, we shall focus on a rescaled (dimensionless) version of the latter system of equations, namely

$$i\frac{\partial\mathcal{E}}{\partial T} + P\frac{\partial^2\mathcal{E}}{\partial Z^2} - Q' N_e\mathcal{E} = 0, \tag{14}$$

where $T = \tau V_{th,e}/\lambda_D$, $Z = \zeta/\lambda_D\tau$, $P = V_g'/(2\lambda_D V_{th,e})$ and $Q' = kV_g\lambda_D/(2V_{th,e})$, and

$$N_e = -\mu\,\frac{V_g^2 - C_{Di}^2}{V_g^2 - C_{DH}^2}(|\mathcal{E}|^2 - |\mathcal{E}_\infty|^2) \equiv \alpha(|\mathcal{E}|^2 - |\mathcal{E}_\infty|^2)\,, \tag{15}$$

where $\mathcal{E} = E_k/\sqrt{4\pi n_{e0}T_e}$, N_e is the electron density perturbation (with respect to equilibrium, as defined above). We note that the proportionality coefficient α bears positive values

inside the interval $V_g \in [C_{DH}, C_{Di}]$ (i.e. for $C_{DH} < V_g < C_{Di}$), and negative values outside. For clarity, by combining all of the above definitions, α can be expressed as

$$\alpha = -\frac{C_{Di}^2 - 4c^2\tilde{k}^2}{\tilde{k}^2[C_{Di}^2 - 4c^2\tilde{k}^2(1 + n_{e0}T_i/n_{i0}T_e)]}\,, \tag{16}$$

where we have defined the dimensionless wavenumber $\tilde{k} = ck\omega_{ce}/\omega_{pe}^2$. Note, once more, the influence of the dust via the ratio n_{e0}/n_{i0}, which takes very small (large) values in a strong presence of negative (positive) dust. Clearly, α takes positive values for $\tilde{k}$ between $\tilde{k}_1 = C_{Di}^2/[4c^2(1+n_{e0}T_i/n_{i0}T_e)]$ and $\tilde{k}_2 = C_{Di}^2/4c^2$, and negative values outside this interval. A few comments may be added here. First, note that $\tilde{k}_1 \approx \tilde{k}_2$ (or $C_{DH} \approx C_{Di}$) for $n_{e0}T_i/(n_{i0}T_e) \ll 1$ (i.e. $\sigma \ll 1$), that is essentially valid in the case of a strong electron depletion due to a high negative charge concentration in the plasma; Eq. (16) reduces to $\alpha \approx \mu = -1/\tilde{k}^2 < 0$ in this case. In the opposite limit $n_{e0}T_i/n_{i0}T_e \gg 1$ (i.e. $\sigma \gg 1$), e.g. in the case of a strong *positive* dust presence, $\tilde{k}_1 \approx 0$ (or $C_{DH} \approx 0$). In any case, let us note that the group velocity V_g is, practically always, above the (slow) dust-acoustic speed C_{Di}, hence prescribing a negative value of α (as well as bright-type envelope excitations, as we shall see below), in a realistic situation.

Combining Eqs. (14) and (15) above, one readily obtains a closed equation in the form of a nonlinear Schrödinger equation (NLSE)

$$i\frac{\partial \mathcal{E}}{\partial T} + P\frac{\partial^2 \mathcal{E}}{\partial Z^2} + Q(|\mathcal{E}|^2 - |\mathcal{E}_\infty|^2)\mathcal{E} = 0\,, \tag{17}$$

where $Q = -Q'\alpha$ and all of the other quantities were defined above. We see that the (constant) asymptotic value of $\mathcal{E}$ at infinity only contributes via a linear term, which may readily be eliminated upon a trivial phase transformation, viz. $\mathcal{E} \to \mathcal{E}' \exp(-iQ|\mathcal{E}_\infty|^2 T)$. Thus, $\mathcal{E}'$ will be understood as the electric field amplitude variable in the following, although the prime will be dropped, for simplicity.

The NLS equation (17) for the electric field amplitude, in combination with Eq. (15) for the electron density perturbation, can be employed in a detailed study of the occurrence of modulational instability of the electron whistlers. According to the standard methodology [10, 11], the latter are *modulationally unstable* (stable) if $PQ > 0$ ($PQ < 0$). To see this, one may first check that the NLSE is satisfied by the plane wave solution $\mathcal{E}(Z,T) = \mathcal{E}_0 \exp(iQ|\mathcal{E}_0|^2 T)$. The standard (linear) stability analysis then shows that a linear modulation (say $\mathcal{E}_0 \to \mathcal{E}_0 + \epsilon\delta\mathcal{E}_0$) with frequency Ω and wavenumber K [i.e.

$\delta\mathcal{E}_0 \sim \exp i(Kx - \Omega t)]$ obeys the dispersion relation

$$\Omega^2 = P\,K^2\,(P\,K^2\ - 2Q\,|\mathcal{E}_0|^2)\,, \tag{18}$$

which exhibits a purely growing unstable mode if $K \leq K_{cr,0} = (2Q/P)^{1/2}\,|\mathcal{E}_0|$ (hence only if $PQ > 0$). The instability growth rate $\sigma = \mathrm{Im}\Omega$ then attains a maximum value $\sigma_{max} = Q\,|\mathcal{E}_0|^2$ at $K_{cr,0}/\sqrt{2}$. For $PQ < 0$, on the other hand, the wave is *stable* to external perturbations.

Combining with our definitions above, we may obtain an exact criterion for whistlers to be modulationally (un)stable. Notice that P and Q are positive quantities, by definition, while Q' yields the opposite sign, with respect to α. Therefore, we see that whistlers will be modulationally *stable* if $\alpha < 0$, i.e. if $C_{DH} < V_g < C_{Di}$ (hence, if $\tilde{k}_1 < \tilde{k} < \tilde{k}_2$; cf. definitions above), and unstable otherwise.

The existence of envelope excitations may be investigated via the above formalism. The NLSE (17) is known [10, 12] to possess localized envelope solutions (envelope solitons) of the bright (dark) type, for $PQ > 0$ ($PQ < 0$). In either case, a localized electric field envelope $\mathcal{E}$ essentially models a modulated wavepacket [viz. $\mathcal{E}\exp(i\theta)$, where $\theta = kz - \omega_k t$ is the fast carrier phase] which is here accompanied by (and co-propagating with) a localized negative electron density variation, i.e. a moving density dip: $N_e = \alpha(|\mathcal{E}|^2 - |\mathcal{E}_\infty|^2) < 0$ (see that $\alpha < 0$ and $\mathcal{E}_\infty = 0 < \mathcal{E}$ in the bright case $PQ > 0$, while $\alpha > 0$ and $0 < \mathcal{E} < \mathcal{E}_\infty$ in the dark case $PQ < 0$; hence, N_e is always negative). For the sake of completeness and future reference, let us briefly summarize the form of the envelope soliton solutions of Eq. (17), omitting unnecessary details regarding their derivation (which can be found e.g. in Ref. [12]). These solutions are obtained upon setting $\mathcal{E} = \mathcal{E}_0\exp(i\Theta)$ in Eq. (17); the amplitude $\mathcal{E}_0$ and the phase correction Θ (both real functions of $\{Z, T\}$) are thus determined. The different types of solution thus obtained are summarized in the following.

For *positive* PQ (i.e. negative α), one obtains the *bright* envelope soliton, i.e. a localized envelope *pulse* of the form

$$\mathcal{E}_0 = \Big(\frac{2P}{QL^2}\Big)^{1/2}\mathrm{sech}\Big(\frac{Z - U_e\,T}{L}\Big)\,, \qquad \Theta = \frac{1}{2P}\Big[U_e Z + \Big(\Omega - \frac{U_e^2}{2}\Big)T\Big]\,, \tag{19}$$

where U_e is the envelope velocity; L and Ω represent the pulse's spatial width and oscillation frequency (at rest), respectively. We note that the soliton width L and maximum amplitude $\hat{\mathcal{E}}_0$ satisfy $L\hat{\mathcal{E}}_0 = (2P/Q)^{1/2} =$ constant. Note that the amplitude $\mathcal{E}_0$ is independent of the pulse (envelope) velocity U_e here.

For $PQ < 0$ (i.e. $\alpha > 0$), the carrier wave is modulationally *stable* and may propagate in the form of a *dark* (*black* or *grey*) envelope wavepacket, i.e. a propagating localized *hole* (a *void*) amidst a uniform wave energy region. The exact expression for the *black* envelope soliton reads

$$\mathcal{E}_0 = \hat{\mathcal{E}}_0 \left| \tanh\Big(\frac{Z - U_e T}{L} \Big) \right| , \qquad \Theta = \frac{1}{2P} \Big[U_e Z + \Big(2PQ\hat{\mathcal{E}}_0^2 - \frac{U_e^2}{2} \Big) T \Big] . \tag{20}$$

Again, $L\mathcal{E}_0 = (2|P/Q|)^{1/2}$ (=cst.).

The *grey*-type envelope (also obtained for $PQ < 0$, or $\alpha > 0$) reads

$$\mathcal{E}_0 \; = \; \hat{\mathcal{E}}_0 \left[1 - d^2 \, \mathrm{sech}^2 \Big(\frac{Z - U_e T}{L} \Big) \right]^{1/2}$$

and

$$\Theta = \frac{1}{2P} \left[V_0 \, Z \; - \Big(\frac{1}{2} V_0^2 - 2PQ\hat{\mathcal{E}}_0^2 \Big) T + \Theta_0 \right] \; - S \, \sin^{-1} \frac{d \, \tanh(\frac{Z - U_e T}{L})}{\left[1 - d^2 \, \mathrm{sech}^2 \Big(\frac{Z - U_e T}{L} \Big) \right]^{1/2}} \, . \tag{21}$$

Here Θ_0 is a constant phase; S denotes the product $S = \mathrm{sign}(P) \times \mathrm{sign}(U_e - V_0)$. The pulse width $L = (2|P/Q|)^{1/2}/(d\hat{\mathcal{E}}_0)$ now also depends on the real parameter d, given by:

$$d^2 \; = \; 1 \; + \; (U_e - V_0)^2/(2PQ\hat{\mathcal{E}}_0^2) \; \leq \; 1 \, .$$

The (real) velocity parameter $V_0 =$ const. satisfies: $V_0 - \sqrt{2|PQ| \, \hat{\mathcal{E}}_0^2} \; \leq \; U_e \; \leq \; V_0 + \sqrt{2|PQ| \, \hat{\mathcal{E}}_0^2}$. For $d = 1$ (thus $V_0 = U_e$), one recovers the *black* envelope soliton.

In all three of the above cases, the electron density variation $N_e = n_{e1}/n_{e0}$ is obtained from Eq. (15) as

$$N_e \; = \; -|\alpha| \, \hat{\mathcal{E}}_0^2 \, \mathrm{sech}^2 \Big(\frac{Z - U_e T}{L} \Big) , \tag{22}$$

where the maximum variation $\hat{\mathcal{E}}_0^2$ and the pulse width L satisfy $\hat{\mathcal{E}}_0^2 L^2 = 2|P/Q|$ (we see that this result also holds in the grey case, regardless of the value of d). We see that the localized density variation is, in fact, a negative (electron depletion) area, whose depth $N_{max} = |\alpha| \, \hat{\mathcal{E}}_0^2$ depends on the plasma parameters essentially via the value of α; it is related to the width L as $N_{max} L^2 = 2|P/Q| =$ constant.

To summarize, we have considered the nonlinear coupling between coherent electron whistlers and ultra-low-frequency DAPs is a magnetized dusty plasma. For that purpose, we have presented a nonlinear Schrödinger equation for a modulated electron whistler wavepacket in the presence of the electron density and magnetic field-aligned electron fluid

perturbations associated with the DAPs. An equation for the latter in the presence of the whistler ponderomotive force was then derived by using the hydrodynamic equations consisting of the electron and dust continuity equations, the inertialess electron and ion momentum equations, and the inertial dust momentum equation. Standard techniques were employed on the governing equations to model the modulational interaction between electron whistlers and non-resonant dust acoustic perturbations. It is shown that modulationally unstable coherent whistlers may form envelope solitons composed of a localized bell-shaped whistler electric field envelope and a density dip/hump created by the whistler ponderomotive force. Dark (black or grey) type envelope excitations, accompanied by a localized density hump, are also possible, yet less likely to be encountered (c.f. the existence conditions in the text).

This investigation may be of relevance to modulated electromagnetic structure observations in the ionosphere. In particular, modulated whistlers accompanied by co-propagating density perturbations are abundantly reported by spacecraft measurements in the magnetosphere. It may be pointed out, for rigor, that the nonlinear amplitude modulation mechanism studied here should be distinguished from the carrier wave self- (auto-)interaction mechanism; c.f. e.g. Ref. [13]. Furthermore, the ponderomotive nonlinear mechanism for wave-plasma interaction described here is distinct from (or, rather, complementary to) the wave heating nonlinearity, studied e.g. in Ref. [14], which may also play an important role at lower ionospheric heights.

Acknowledgments

Support from the *Deutsche Forschungsgemeinschaft* through *Sonderforschungsbereich (SFB) 591 – Universelles Verhalten Gleichgewichtsferner Plasmen: Heizung, Transport und Strukturbildung* is also gratefully acknowledged.

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