Discrete breather modes associated with vertical dust grain oscillations in dusty plasma crystals

I. Kourakis[a] and P. K. Shukla[b]
Institut für Theoretische Physik IV, Fakultät für Physik und Astronomie, Ruhr-Universität Bochum, D-44780 Bochum, Germany

(Received 8 June 2004; accepted 27 September 2004; published online 3 December 2004)

Accounting for the lattice discreteness and the sheath electric field nonlinearity in dusty plasma crystals, it is demonstrated that highly localized structures (discrete breathers) involving vertical (transverse, off-plane) oscillations of charged dust grains may exist in a dust lattice. These structures correspond to either extremely localized bright breather excitations (pulses) or dark excitations composed of dips/voids. Explicit criteria for selecting different breather modes are presented. © *2005 American Institute of Physics*. [DOI: 10.1063/1.1824908]

Recent studies of collective processes in a dust-contaminated plasma (DP) (Ref. 1) have revealed a variety of different linear and nonlinear collective effects, which are observed in laboratory and space dusty plasmas. An issue of particular importance in DP research is the formation of strongly coupled DP crystals by highly charged dust grains, typically in the sheath region above a horizontal negatively biased electrode in experiments.[1,2] Typical low-frequency oscillations are known to occur[2] in these mesoscopic dust grain quasilattices in the longitudinal (in-plane, acoustic mode), horizontal transverse (in-plane, shear mode) and vertical transverse (off-plane, opticlike mode) directions.

It is known from solid state physics that periodic lattices may sustain, a variety of localized excitations, due to a mutual balance between the intrinsic nonlinearity of the medium and the mode dispersion. Such structures, typically investigated in a continuum approximation (i.e., assuming that the typical spatial variation scale far exceeds the typical lattice scale, e.g., the lattice constant r_0), include nontopological *solitons* (pulses), *kinks* (i.e., shocks or dislocations), and localized modulated envelope structures (*envelope solitons*), and generic nonlinear theories have been developed in order to investigate their relevance in different physical contexts.[3] In addition to these (continuum) theories, attention has been paid since more than a decade ago to highly localized (either propagating or stationary) vibrating structures [e.g., *discrete breathers* (DBs) which are also widely referred to as *intrinsic localized modes*], which owe their very existence to the lattice discreteness itself. Thanks to a few pioneering works[4–8] and a number of studies which followed, many aspects involved in the spontaneous formation, mobility and interaction of DBs are now elucidated, both theoretically and experimentally; see in Refs. 9–11 for a review.

Admittedly, even though nonlinearity is an inherent feature of the dust crystal dynamics (either due to intergrain electrostatic interactions, to mode coupling or to the sheath environment, which is intrinsically anharmonic), the present day knowledge of nonlinear mechanisms related to dust lattice modes is still in a preliminary stage. Small amplitude localized longitudinal excitations (described by a *Boussinesq* equation for the longitudinal grain displacement u, or a *Korteweg–de Vries* equation for the density $\partial u/\partial x$) were considered in Refs. 12 and 13 and generalized in Ref. 14. Also, the amplitude modulation of longitudinal[15,16] and transverse (vertical, off-plane) (Refs. 17 and 18) dust lattice waves was recently considered. As a matter of fact, all of these studies rely on a quasicontinuum description of the dust lattice. However, the effect of the high discreteness of dust crystals, clearly suggested by experiments,[19,20] may play an important role in mechanisms such as energy localization, information storage, and response to external excitations, in view of DP application design, in particular. In this paper a study is carried out, from first principles, of the relevance of DB excitations with respect to dust lattice waves.

Let us consider the vertical (off-plane) grain displacement in a dust crystal (assumed quasi-one-dimensional and infinite: identical grains of charge q and mass M are situated at $x_n = nr_0$, where $n = \ldots, -1, 0, 1, 2, \ldots$), taking into account the intrinsic nonlinearity of the sheath electric (and/or magnetic) potential. The vertical grain displacement obeys an equation of the form

$$d^2\delta z_n/dt^2 + \nu(d\delta z_n/dt) + \omega_0^2(\delta z_{n+1} + \delta z_{n-1} - 2\delta z_n) + \omega_g^2\delta z_n + \alpha(\delta z_n)^2 + \beta(\delta z_n)^3 = 0, \quad (1)$$

where $\delta z_n = z_n - z_0$ denotes the small displacement of the nth grain around the (levitated) equilibrium position z_0, in the transverse (z) direction. The characteristic frequency $\omega_0 = [-q\Phi'(r_0)/(Mr_0)]^{1/2}$ results from the dust grain (electrostatic) interaction potential $\Phi(r)$, e.g., for a Debye–Hückel potential:[21] $\Phi_D(r) = (q/r)e^{-r/\lambda_D}$, one has $\omega_{0,D}^2 = q^2/(Mr_0^3)(1 + r_0/\lambda_D)\exp(-r_0/\lambda_D)$, where λ_D denotes the effective DP Debye radius.[1] The damping coefficient ν accounts for dissipation due to collisions between dust grains and neutral atoms. The gap frequency ω_g and the nonlinearity coefficients α, β are defined via the overall vertical force $F(z) = F_{e/m} - Mg$

[a]On leave from U.L.B.—Université Libre de Bruxelles, Physique Statistique et Plasmas, C. P. 231, Boulevard du Triomphe, B-1050 Brussels, Belgium. Also at Faculté des Sciences Apliquées, C.P. 165/81, Physique Générale, Avenue F. D. Roosevelt 49, B-1050 Brussels, Belgium. Electronic mail: ioannis@tp4.rub.de
[b]Electronic mail: ps@tp4.rub.de

$\approx -M[\omega_g^2 \delta z_n + \alpha(\delta z_n)^2 + \beta(\delta z_n)^3] + O[(\delta z_n)^4]$, which has been expanded around z_0 by formally taking into account the (anharmonicity of the) local form of the sheath electric (follow exactly the definitions in Ref. 17, not reproduced here) and/or magnetic[22] field(s), as well as, possibly, grain charge variation due to charging processes.[18] Recall that the electric/magnetic levitating force(s) $F_{e/m}$ balance(s) gravity at z_0. Notice the difference in structure from the usual nonlinear Klein–Gordon equation used to describe one-dimensional oscillator chains—cf., e.g., Eq. (1) in Ref. 6: "phonons" in this chain are stable only in the presence of the field force $F_{e/m}$. Collisions with neutrals and coupling anharmonicity are omitted, at a first step, in this simplified model.

We may now neglect neutral-dust collisions, i.e., setting $\nu \to 0$, for simplicity; the dissipative case, to be discussed below, will be addressed in a more detailed future work. Retaining only the linear contribution and considering phonons of the type, $\delta z_n \sim \exp[i(knr_0 - \omega t)]$+ complex conjuguate (c.c.), one obtains[1] the well known transverse dust lattice (TDL) wave, whose frequency ω is determined from

$$\omega^2 = \omega_g^2 - 4\omega_0^2 \sin^2(kr_0/2), \tag{2}$$

where $k = 2\pi/\lambda$ denotes the wave number. We observe that the wave frequency ω decreases with increasing k (or decreasing wavelength λ), implying that the transverse vibrations propagate as a *backward wave*: the group velocity $v_g = \omega'(k)$ and the phase speed $v_{ph} = \omega/k$ have opposite directions (this is in agreement with recent experiments[20]). The modulational stability profiles of these linear waves (depending on the plasma parameters) have been investigated in Refs. 17 and 18. In Eq. (2) we notice the natural *gap frequency* ω_g, corresponding to an overall motion of the chain's center of mass, as well as the *cutoff frequency* $\omega_{\min} = (\omega_g^2 - 4\omega_0^2)^{1/2}$ (obtained at the end of the first Brillouin zone $k = \pi/r_0$) which is *absent in the continuum limit*, viz., $\omega^2 \approx \omega_g^2 - \omega_0^2 k^2 r_0^2$ (for $k \ll r_0^{-1}$).

We consider the (first harmonic) amplitude of the dust grain oscillations to be small, say of order ϵ ($0 < \epsilon \ll 1$; ϵ is a small parameter). We keep full account of the lattice discreteness, by assuming that the dust grain oscillations may be characterized by a *strong* variation from one site to another. Physically, this implies that isolated dust grain oscillations (at the characteristic frequency ω_g) far exceed in strength the intergrain coupling (typically measured by ω_0). According to these considerations, and following the treatment suggested in Ref. 6, one may adopt the ansatz $\delta z_n \approx \epsilon(u_{1,n}^{(1)} e^{-i\omega_g t} + \text{c.c.}) + \epsilon^2[u_{2,n}^{(0)} + (u_{2,n}^{(2)} e^{-2i\omega_g t} + \text{c.c.})] + \cdots$, in combination with the "book-keeping" scaling assumption: $\omega_g^2, \alpha, \beta \sim 1$ and $d/dt, \omega_0^2 \sim \epsilon^2$. The physical meaning of these assumptions is clear: the coupling between dust grains (measured by $\omega_0 \approx \epsilon\omega_g$) is expected to affect the grain dynamics via the generation, to order ϵ^2, of 2nd and 0th order harmonics, drawing inspiration from the insight gained from the solution of the continuum problem.[17] The conditions underlying this hypothesis, which are actually in agreement with experiments (see the *narrow* frequency bands obtained in Refs. 19 and 20, which suggest a high ω_g/ω_0 ratio), are (intuitively speaking) better fulfilled for higher values of the lattice parameter $\kappa = r_0/\lambda_D$ (hence weaker coupling) and/or for stronger levitation fields (i.e., higher ω_g).

Inserting the above ansatz into Eq. (1) and keeping terms up to order ϵ^3, one obtains a discrete nonlinear Schrödinger-type equation of the form

$$i(du_n/dt) + P(u_{n+1} + u_{n-1} - 2u_n) + Q|u_n|^2 u_n = 0, \tag{3}$$

where $u_n = u_{1,n}^{(1)}(t)$, along with the harmonic amplitude relations

$$u_{2,n}^{(2)} = (\alpha/3\omega_g^2)u^2, \quad u_{2,n}^{(0)} = -(2\alpha/\omega_g^2)|u|^2, \tag{4}$$

and the definitions

$$P = -\omega_0^2/(2\omega_g), \quad Q = (10\alpha^2/3\omega_g^2 - 3\beta)/2\omega_g \tag{5}$$

for the discreteness and nonlinearity coefficients P and Q, respectively.[23] We note that $P < 0$. The sign of Q, on the other hand, depends on the sheath characteristics and cannot be prescribed.

The envelope equation (3) yields a plane wave solution $u_n = u_0 \exp(i\theta_n) \equiv u_0 \exp[-i(n\tilde{k}r_0 - \tilde{\omega}t)]$+c.c., where the envelope frequency $\tilde{\omega}$ obeys the dispersion relation

$$\tilde{\omega}(\tilde{k}) = -4P \sin^2(\tilde{k}r_0/2) + Q|u_0|^2. \tag{6}$$

Notice that the *nonlinear envelope dispersion relation* (6) prescribes the existence of two distinct (envelope) frequency *cutoffs*, since the permitted frequency values will lie in the band between $\tilde{\omega}_{cr,1} = \tilde{\omega}(\tilde{k}=0) = |Q||u_0|^2$ and $\tilde{\omega}_{cr,2} = \tilde{\omega}(\tilde{k}=\pi/2) = |Q|u_0|^2 - 4P|$; recall that $P < 0$ (Ref. 24) (the latter cutoff is absent in the continuum theory). Now, perturbing both the amplitude u_0 and the phase θ_n, i.e., setting $u_0 \to u_0 + \xi\hat{u}_0 \exp[-i(n\hat{k}r_0 - \hat{\omega}t)]$ and $\theta_n \to \theta_n + \xi\hat{\theta}_0 \exp[-i(n\hat{k}r_0 - \hat{\omega}t)]$, where $\hat{k}, \hat{\omega} \in \Re$ and $\xi \ll 1$, and then linearizing in ξ, one obtains the *perturbation* dispersion relation

$$[\hat{\omega} - 2P \sin \hat{k}r_0 \sin \tilde{k}r_0]^2 = 4P \sin^2(\hat{k}r_0/2)\cos \tilde{k}r_0[4P \sin^2(\hat{k}r_0/2)\cos \tilde{k}r_0 - 2Q|u_0|^2], \tag{7}$$

which determines the stability of a plane wave with the envelope wave number $\tilde{k}$ propagating in the lattice. Contrary to the continuum modulated envelope limit case, readily recovered from Eq. (7) for $\hat{k}r_0$, $\tilde{k}r_0 \ll 1$, where stability would be prescribed for $PQ < 0$ (cf. Ref. 17), the modulational stability profile now also depends on $\tilde{k}$: instability (viz., Im $\hat{\omega} \neq 0$) occurs if $PQ \cos \tilde{k}r_0 > 0$ (i.e., essentially $Q \cos \tilde{k}r_0 < 0$ here) *and* (at the same time) the amplitude u_0 exceeds some $\tilde{k}$- dependent critical value u_{cr} [or, in fact, for any $\tilde{k}$ if $u_0 > (2P/Q)^{1/2} = u_{cr,\max}$].

It is known that the modulational instability, possible, e.g., for $PQ > 0$ (viz., $Q < 0$ here) *and* $0 < \tilde{k} < \pi/2r_0$, in our model, may result in the formation of localized modes [i.e., solutions of Eq. (3)] in the linear frequency gap region[26] (i.e. the range of "forbidden" frequency values below the phonon dispersion curve). Following previous works,[4,6] *odd-parity* localized periodic solutions may be sought in the form $u_n(t) = f_n u_0 \cos \Omega t = f_n u_0/2(\exp i\Omega t + \text{c.c.})$, assuming ($u_0$

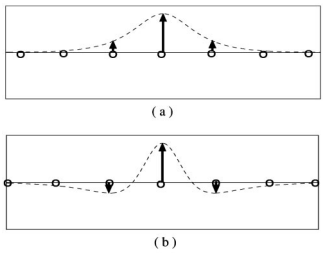

FIG. 1. Localized discrete breather lattice excitations of the bright type (i.e., with vanishing displacement at infinity), obtained for $Q<0$; the successive lattice site displacements are depicted at maximum amplitude: (a) odd-parity solution, as given by Eq. (8); (b) even-parity solution, as given by Eq. (9).

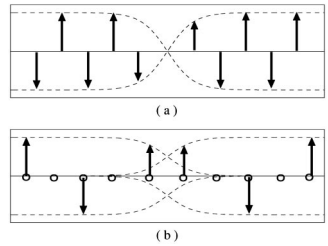

FIG. 2. Localized discrete breather lattice excitations of the dark type (i.e., with a nonvanishing displacement at infinity), obtained for $Q>0$ (successive lattice site displacements): (a) dark-type solution, as given by (10); (b) gray-type solution (in the middle of the Brillouin zone), as given by (13).

$\in$ Re and) $f_0=1$, $f_{-n}=f_n$ and $|f_n|\ll f_1$ for $|n|>1$. Combining the equations obtained from Eq. (3) for $n=0$ and $n=1$, one obtains a localized lattice excitation, for $u_n(t)$, which follows the pattern,

$$u_0 \cos \Omega t \times \{\ldots,0,0,\eta,1,\eta,0,0,\ldots\}, \tag{8}$$

for $n\in\{\ldots,-2,-1,0,1,2,\ldots\}$, respectively, where $\Omega \approx |Q|u_0^2/4$ and $\eta=4P/(Qu_0^2)\ll 1$ ($\eta>0$). We see that Ω lies outside the (amplitude) frequency band prescribed by (6).

For $PQ<0$ (i.e., $Q>0$, since $P<0$), the modulational stability profile is reversed: instability occurs for $\tilde{k}>\pi/2r_0$ (viz., $\cos \tilde{k}r_0<0$), while excitations with lower $\tilde{k}$ (i.e., $\lambda>4r_0$) will be stable. The same procedure then leads to a lattice structure [*even parity* solution for $u_n(t)$], which follows the pattern,

$$u_0 \cos \Omega t \times \{\ldots,0,0,-\eta,1,-\eta,0,0,\ldots\}, \tag{9}$$

where $\eta=-P/(Qu_0^2)$ satisfies $0<\eta\ll 1$. Again, the DB frequency $\Omega\approx -4P+Qu_0^2$ lies outside the "phonon" band prescribed by Eq. (6).

The excitations described above, which are depicted in Fig. 1, can be shown to be remarkably long lived, either by employing simple, heuristic arguments (as, e.g., suggested in Ref. 6) or via a more rigorous mathematical treatment (to be addressed in a future work), ensuring that the decay (if any) of these DBs into radiation in the dust crystal (phonons) will be exponentially slow. Remarkably, this results also holds in the forced and damped (dissipative) general case; see the discussion below. An interested reader may see Refs. 9 and 10 for theoretical details.

It is well known[18] that in the continuum limit Eq. (3) possesses bright envelope solutions (pulses) for $PQ>0$, as well as dark/gray-type excitations obtained for $PQ<0$. The former (bright) solutions (directly analogous in form, for high lattice discreteness, to the above ones) are known to be spontaneously formed as a result of the wave amplitude's (modulational) instability. On the other hand, the latter (dark) solutions are encountered in the region where the carrier wave is *stable* and are identified as localized wave energy dips (*voids*) against an otherwise uniform wave energy background (see, e.g., Figs. 3 and 4 in Ref. 18).

Inspired by the continuum limit, one might find it meaningful to seek such solutions in our discrete dust lattice, in the TDL carrier wave stable wave number region, i.e., where $PQ\cos \tilde{k}r_0<0$ (or $Q\cos \tilde{k}r_0>0$; see above). *For negative* Q, for instance, one should look into the region $\tilde{k}>\pi/(2r_0)$, e.g., near the cutoff frequency $\tilde{\omega}_{cr,2}=4P-Q|u_0|^2$. Since neighboring sites are known to oscillate out of phase near this edge of the Brillouin zone, one may seek localized solutions by substituting in Eq. (3) with $u_n=(-1)^n\Psi(x,t)\times\exp(-i\Omega t)+\text{c.c.}$, where $\Psi(x,t)$ is here taken to be a continuum function of site position $x=nr_0$ and time t, for the sake of analytical tractability. A straightforward calculation then provides the solution $u_n=(-1)^n A\tanh(A\sqrt{P}nr_0)\times\cos\Omega t$, where the envelope frequency is $\Omega=4P-Q|A|^2$. Similar dark-type discrete chain patterns may be sought in the form

$$u_n(t) = 2A\cos\Omega t\{\ldots,1,-1,1,-(1-\eta),0,1-\eta,-1,1,-1,\ldots\}, \tag{10}$$

where $\Omega=4P-Q|A|^2$ and $\eta=P/(QA^2)$ is a small number satisfying $0<\eta\ll 1$. This dark-type discrete lattice excitation[6] is modulated by two opposite narrow shocklike (kink) excitations [see Fig. 2(a)]. This is, in fact, an approximate solution of (3), which fits more accurately for lower values of η.

An interesting phenomenon occurs in the middle of the Brillouin zone, i.e., at $k=\pi/2r_0$, where standard solid state theory prescribes a twofold behavior of the discrete lattice: *odd* (or even) sites will remain immobile, while *even* (or odd, respectively) ones will oscillate in an alternating phase fashion at a frequency $\omega_1=(\omega_g^2-2\omega_0^2)^{1/2}$. This particular motion was exhaustively analyzed in Ref. 25 for an even substrate potential, where a different type of discrete lattice excitation of *gray type* (voidlike, but with a nonvanishing amplitude) was proposed.

Given the above described topology of the expected lattice pattern, one may seek a solution in the form of a superposition of two different excitations, viz., $u_{2m}\equiv v_{2m}=(-1)^m V(2mr_0,t)\exp(i\omega_1 t)$ and $u_{2m+1}\equiv w_{2m+1}=(-1)^m W[(2m+1)r_0,t]\exp(i\omega_1 t)+\text{c.c.}$, respectively, where the functions V and W are slowly varying in space. Assuming that $\{V,W\}=\{f_1,f_2\}\exp(i\Omega t)$, where $f_1,f_2,\Omega\in\Re$, one obtains

$$f_2^2=\frac{\Omega\exp[\mp\Omega x/(\sqrt{2}Pr_0)]\{\cosh[\Omega x/(\sqrt{2}Pr_0)]\pm\sqrt{2}\}}{2Q\cosh[\sqrt{2}\Omega x/(Pr_0)]}, \tag{11}$$

and

$$f_1 = \exp[\pm\Omega x/(\sqrt{2}Pr_0)]f_2, \tag{12}$$

which represent a kink-antikink pair, i.e., a superposition of two opposite polarity shocklike excitations: one with zero ($f_0=\sqrt{\Omega/Q}$) asymptotic values at $-\infty$ ($+\infty$), and its inverse, but with a shifted origin (hence the nonvanishing value at $x=0$); see the dashed curves in Fig. 2(b). Now, looking for similar-looking discrete lattice patterns, one finds the form[6]

$$u_n(t) = 2A\cos\Omega t\{\ldots,1,0,-1,0,1-\eta,1-\eta,0,-1,0,1,\ldots\}, \tag{13}$$

where $\Omega=2P-Q|A|^2$ is the frequency in the middle of the Brillouin zone and $\eta=P/2QA^2$ is a small number satisfying $0<\eta\ll 1$. This gray-type discrete lattice excitation is, again, an approximate solution of Eq. (3). It may be noted that the existence of this kind of (double kink, "*noncutoff*") solution has been investigated (and confirmed) experimentally in a chain of coupled pendulums,[27] which obeys an equation similar (yet not identical) to our Eq. (1).

Reference should be made, for the sake of rigor, to *dissipative* (*damping*) or external *forcing* effects, which were omitted while deriving the above solutions. This is indeed expected to be an important issue in real dust crystals, where dust-neutral collisions, ion drag, and external fields are known to share their part in the dynamics. In condensed matter physics, issues regarding DBs (existence, mobility, interaction, sustainance, decay, …) have been addressed exhaustively in the last decade, both analytically and numerically. In brief, DBs were shown to exist in a *conservative* system, provide that their frequency (or its multiples) enter in no resonance with the (linear) excitation frequency band; obviously, this condition is easier to satisfy if the dispersion curve is rather flat (i.e., if the permitted linear wave frequency band is sufficiently narrow) and, most relieving, TDL waves in dust crystals are indeed characterized by such a dispersion (see in Refs. 19–21). On the other hand, in a dissipative and forced system, theoretical predictions[28–30] suggest that the intrinsic localization of energy is *not* destroyed by resonances, due to the efficient damping of radiation away from the localization site (center of the breather); instead, dissipative DBs act as a generic *attractor* in the dynamical phase space of the system in this case.[30] The nonresonance constraint mentioned above is then obsolete (note that the linear wave frequency range may be broader due to damping). As a conclusion, the *existence*, *localization* (spatial decay is exponential, in general), *sustainance* (for sufficiently long times), and *robustness* (with respect to random external perturbations, noise, thermostats) of *breathers* appear to be established in both conservative and dissipative discrete systems.

To summarize, we have presented the possibility of the occurrence of localized discrete breather-type excitations associated with vertical dust grain oscillations in a dust monolayer. We have shown that a dusty plasma crystal can sustain either bright- or dark-/gray-type excitations, i.e., extremely localized pulses or voids, respectively. The localized structures presented here owe their existence to the intrinsic lattice discreteness in combination with the nonlinearity of the plasma sheath. Both are experimentally tunable physical mechanisms, so our results may be investigated (and will hopefully be verified) by appropriately designed experiments, which should in principle show the way to potential applications relying on pulse localization in dusty plasma crystals. Finally, the question of the existence of multimode breathers and/or multibreathers, i.e., discrete excitations in the form: $u_n(t)=\Sigma_{k=-\infty}^{\infty}A_n(k)\exp(ik\omega t)$ [with $A_n(k)=A_n^*(-k)$ for reality and $|A_n(k)|\to 0$ as $n\to\pm\infty$, for localization], and also in the presence of energy damping and forcing, will be more rigorously addressed in a forthcoming work, which is in preparation.

I.K. wishes to thank S. Flach (MPIPKS, Dresden, Germany) and V. Koukouloyannis (AUTh, Thessaloniki, Greece) for a number of elucidating discussions.

This work was partially supported by the Deutsche Forschunsgemeinschaft through Sonderforschungsbereich Grant No. 591—Universelles Verhalten gleichgewichtsferner Plasmen: Heizung, Transport und Strukturbildung.

[1] P. K. Shukla and A. A. Mamun, *Introduction to Dusty Plasma Physics* (Institute of Physics, Bristol, 2002).
[2] G. E. Morfill, H. M. Thomas, and M. Zuzic, in *Advances in Dusty Plasma Physics*, edited by P. K. Shukla, D. A. Mendis, and T. Desai (World Scientific, Singapore, 1997), p. 99.
[3] M. Remoissenet, *Waves Called Solitons* (Springer, Berlin, 1994).
[4] S. Takeno, K. Kisoda, and A. J. Sievers, Prog. Theor. Phys. Suppl. **94**, 242 (1988); J. B. Page, Phys. Rev. B **41**, 7835 (1990).
[5] T. Dauxois and M. Peyrard, Phys. Rev. Lett. **70**, 3935 (1993).
[6] Yu. Kivshar, Phys. Lett. A **173**, 172 (1993).
[7] R. S. McKay and S. Aubry, Nonlinearity **7**, 1623 (1994).
[8] S. Flach, C. R. Willis, and G. Mutschke, Phys. Rev. E **49**, 5018 (1994).
[9] S. Flach, and C. R. Willis, Phys. Rep. **295**, 181 (1998).
[10] See various articles in the Volume (Focus Issue): edited by Yu. Kivshar and S. Flach, Chaos **13**, 586 (2003).
[11] D. K. Campbell, S. Flach, and Yu. S. Kivshar, Phys. Today **57**, 43 (2004).
[12] F. Melandsø, Phys. Plasmas **3**, 3890 (1996).
[13] P. K. Shukla and A. A. Mamun, New J. Phys. **5**, 17 (2003).
[14] I. Kourakis and P. K. Shukla, Eur. Phys. J. D **29**, 247 (2004).
[15] M. R. Amin, G. E. Morfill, and P. K. Shukla, Phys. Plasmas **5**, 2578 (1998); Phys. Scr. **58**, 628 (1998).
[16] I. Kourakis and P. K. Shukla, Phys. Plasmas **11**, 1384 (2004).
[17] I. Kourakis and P. K. Shukla, Phys. Plasmas **11**, 2322 (2004).
[18] I. Kourakis and P. K. Shukla, Phys. Plasmas **11**, 3665 (2004).
[19] B. Liu, K. Avinash, and J. Goree, Phys. Rev. Lett. **91**, 255003 (2003).
[20] T. Misawa, N. Ohno, K. Asano, M. Sawai, S. Takamura, and P. K. Kaw, Phys. Rev. Lett. **86**, 1219 (2001).
[21] U. Konopka, G. E. Morfill, and L. Ratke, Phys. Rev. Lett. **84**, 891 (2000).
[22] In the *magnetically* levitated dust crystal case, the reader should consider the definitions in Ref. 18, upon setting $K_1\to\alpha$, $K_2\to\beta$, and $K_3\to 0$ therein.
[23] Equations (3)–(5) generalize previous results (Ref. 17) obtained in the quasicontinuum limit (i.e., assuming a continuum envelope but discrete carrier wave dynamics).
[24] The absolute value of the right-hand side in Eq. (6) may just as well be considered instead, due to the physical interpretation of $u_n\in\Re$.
[25] Yu. Kivshar, Phys. Rev. B **46**, 8652 (1992).
[26] Yu. Kivshar and M. Peyrard, Phys. Rev. A **46**, 3198 (1992).
[27] B. Denardo, B. Galvin, A. Greenfield, A. Larraza, S. Putterman, and W. Wright, Phys. Rev. Lett. **68**, 1730 (1992).
[28] S. Aubry, Physica D **103**, 201 (1997).
[29] J.-A. Sepulchre and R. S. McKay, Nonlinearity **10**, 679 (1997).
[30] R. S. McKay and J.-A. Sepulchre, Physica D **119**, 148 (1998).