PHYSICS OF PLASMAS VOLUME 11, NUMBER 9 SEPTEMBER 2004

Nonlinear Lagrangian theory of envelope electrostatic plasma waves in a two-electron-temperature plasma

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(Received 12 March 2004; accepted 15 June 2004; published online 26 August 2004)

An analytical model based on Lagrangian variables is presented for the description of ion-acoustic waves propagating in an unmagnetized, collisionless, three-component plasma composed of inertial positive ions and two thermalized electron populations, characterized by different temperatures. The wave's amplitude is shown to be modulationally unstable. Different types of localized envelope electrostatic excitations are shown to exist, and their forms are analytically and numerically investigated in terms of the plasma dispersion and nonlinearity laws. These results are in qualitative agreement with satellite observations in the magnetosphere. © *2004 American Institute of Physics*. [DOI: 10.1063/1.1781167]

I. INTRODUCTION

Ion acoustic waves (IAWs) are well-known electrostatic plasma modes,[1] excited when a population of inertial ions oscillate against a dominant thermalized background of electrons providing the necessary restoring force. The IAW phase speed lies between the electron and ion thermal speeds.

The linear properties of the IAWs have been extensively studied and have been well understood for a long time.[1] As far as nonlinear effects are concerned, the formation of IAW-related localized structures, due to a mutual compensation between nonlinearity and dispersion, has been anticipated theoretically, either via the Korteweg–deVries (KdV) or Zakharov-Kuznetsov equation,[2–4] describing small amplitude solitary waves, or the Sagdeev potential formalism,[4–7] accounting for arbitrary amplitude excitations, and also experimentally confirmed.[8]

It is now established that the characteristics of the IAW propagation can be strongly modified by the existence of a minority population of "cold" electrons, as has been shown theoretically[3,4,6,9,10] and experimentally.[9,11] Regarding applications, it may be noted that the injection of cold electrons into a plasma and the subsequent decrease of the phase speed of waves has been suggested as a possible IAW stabilization mechanism, as well as an ion heating enhancement method.[9]

Recently, studies of two-electron-temperature plasmas have been encouraged by satellite observations of moving localized potential variation regions, reported by recent spacecraft missions, e.g., the FAST (Fast Auroral Snapshot) in the auroral region,[12,13] as well as the S3-3,[14] Viking,[15,16] GEOTAIL, and POLAR (Ref. 13 and 17) missions in the Earth's magnetosphere, where such an electron population coexistence is encountered. Some of the localized structures reported therein bear qualitative characteristics which are reminiscent of electrostatic solitary waves and are strongly believed to be related to ion-acoustic waves; see the discussion in Ref. 13; also see Ref. 18 for a recent consideration of ion-acoustic shocks with respect to the Auroral Kilometric Radiation. It should be stressed that both compressive and rarefactive large amplitude structures have been observed;[12] as a matter of fact, it has recently been suggested[13] that neither the velocity dependence of the observed potential structure amplitudes nor their asymmetry should be taken for granted, since they may be attributed to intrinsic measurement errors; finally, the observed phase speeds lie over an extended region of values, sometimes even above the ion sound speed; these facts seem to suggest that plainly employing the KdV picture[4,10,18] may not suffice for the elucidation of the generation of these solitary structures and an alternative instability mechanism may be present; also see the discussion in Refs. 10 and 13. One such mechanism, namely, the modulational instability of the IAW due to carrier wave self-interaction, was suggested in Ref. 19 where it was shown to lead to harmonic generation and the formation of envelope localized structures of either compressive or rarefactive form, in agreement with satellite observations.[20,21] Finally, IAWs were recently studied via an approximative approach relying on a Lagrangian variable formulation,[22] based on a formalism previously employed in the description of electron plasma waves[23–25] and thoroughly studied in a recent review paper[26] with respect to plasma and fluid flows. Our aim here is to extend those results by applying the Lagrangian formalism to the description of nonlinear IAWs in plasmas with two-electron components.

In this paper, we consider the nonlinear propagation of ion-acoustic waves in a collisionless plasma consisting of three distinct particle species "*s*": an inertial species of ions (denoted by "*i*"; mass m_i, charge $q_i = +Z_i e$), surrounded by an environment of two populations of ("hot" and "cold") elec-

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trons (mass m_e, charge $-e$), at different temperatures T_h and $T_c \equiv \mu T_h < T_h$. Charge neutrality is assumed at equilibrium.

II. THE MODEL

Let us consider the hydrodynamic-Poisson system of equations for the IAWs in the plasma. The number density n_i of ions is governed by the continuity equation

$$\frac{\partial n_i}{dt} + \nabla \cdot (n_i \mathbf{u}_i) = 0, \tag{1}$$

where the mean velocity $\mathbf{u}_i$ obeys

$$\frac{\partial \mathbf{u}_i}{dt} + \mathbf{u}_i \cdot \nabla \mathbf{u}_i = \frac{Z_i e}{m_i}\mathbf{E} = -\frac{Z_i e}{m_i}\nabla \Phi. \tag{2}$$

The electric field $\mathbf{E}=-\nabla\Phi$ is expressed in terms of the gradient of the potential Φ, which is obtained from Poisson's equation $\nabla\cdot\mathbf{E}=4\pi\Sigma q_s n_s$, viz.,

$$\nabla^2\Phi = 4\pi e(n_c + n_h - Z_i n_i). \tag{3}$$

Alternatively, one may consider

$$\frac{\partial \mathbf{E}}{\partial t} = -4\pi \sum q_s n_s \mathbf{u}_s. \tag{4}$$

We assume a near-Boltzmann distribution for both electron populations, i.e., $n_* \approx n_{*,0}\exp(e\Phi/k_B T_*)$ (T_* is the hot/cold electron temperature for $*=h/c$; k_B is Boltzmann's constant; obviously $u_{h/c}=0$ here). The overall quasineutrality condition reads

$$n_{c,0} + n_{h,0} - Z_i n_{i,0} = 0. \tag{5}$$

One-dimensional propagation along the magnetic field $\mathbf{B}$ is explicitely considered in the following (so ∇ will henceforth denote $\partial/\partial x$), thus we have neglected the Lorentz forces in Eq. (2).

A. Reduced Eulerian description

Choosing appropriate physical scales, one may cast all of these equations into a reduced (nondimensional) form. First, let us define the effective electron temperature $T_{eff} = (n_{h,0}+n_{c,0})/(n_{h,0}/T_h + n_{c,0}/T_c)$ and the "effective sound speed" $c_{s,eff}=(k_B T_{eff}/m_i)^{1/2}$, in agreement with previous experimental[9] and theoretical[10] considerations. Notice that, interestingly, the effective electron temperature T_{eff} remains finite, even if the hot electron temperature T_h tends to infinity (in the presence of even a small percentage of cold electron population) as pointed out in Ref. 9. The space and time scales, L and $T=L/c_{s,eff}$, are chosen as the effective Debye length $\lambda_{D,eff}=(k_B T_{eff}/4\pi Z_i^2 n_{i,0} e^2)^{1/2} \equiv c_{s,eff}/\omega_{p,i}$ and the inverse ion plasma frequency $\omega_{p,i}^{-1}=(4\pi n_{i,0} Z_i^2 e^2/m_i)^{-1/2}$, respectively.

Equations (1)–(3) can thus be combined into the reduced equations

$$\frac{\partial n}{dt} + \nabla (n\, u) = 0,$$

$$\frac{\partial u}{dt} + u\,\nabla u = -\nabla \phi,$$

and

$$\nabla^2 \phi = -(n - \hat{n}), \tag{6}$$

where all quantities are nondimensional: $n=n_i/n_{i,0}$, $\mathbf{u}=\mathbf{u}_i/v_0$, and $\phi=\Phi/\Phi_0$; the scaling quantities are, respectively, the equilibrium ion density $n_{i,0}$, the effective sound speed $v_0=c_{s,eff}$ (defined above), and $\Phi_0=k_B T_{eff}/(Z_i e)$. The (reduced) electron background density $\hat{n}$ is defined as

$$\hat{n} = \frac{n_{c,0}}{Z_i n_{i,0}} e^{(T_{eff}/Z_i T_c)\phi} + \frac{n_{h,0}}{Z_i n_{i,0}} e^{(T_{eff}/Z_i T_h)\phi} \equiv \sum_{s=c,h} N_s e^{\beta_s \phi}, \tag{7}$$

where obviously $N_s=n_{s,0}/(Z_i n_{i,0})$ and $\beta_s=T_{eff}/(Z_i T_s)$ (for $s=c,h$). We note that both n and $\hat{n}$ reduce to unity at equilibrium [since Eq. (5) implies $N_c+N_h=1$]. Notice, for later reference, that an approximate expression is obtained for $\hat{n}$ by explicitly assuming that $T_c \ll T_h$, i.e., $\mu=T_c/T_h=\beta_h/\beta_c \ll 1$, and then making use of the neutrality condition (5); one thus obtains the expression

$$\hat{n} \approx N_c(e^{\beta_c\phi}-1) + 1 \equiv f(\phi). \tag{8}$$

Let us define, for later use, the inverse function of f,

$$f^{-1}(x) = \frac{1}{\beta_c}\ln\left(1 - \frac{1-x}{N_c}\right) \equiv g(x), \tag{9}$$

viz., $f(\phi)=x$ implies $\phi=f^{-1}(x)\equiv g(x)$.

Retain the definition of the cold-to-hot electron density and temperature ratio, namely, $\nu=n_{c,0}/n_{h,0}$ and $\mu=T_c/T_h$. Notice that, taking the isothermal limit $\nu\to 0$ (or $\nu\to\infty$ or $\mu\to 1$), one recovers exactly the "ordinary" IAW (in electron-ion plasma) case. The parameters ν and μ generally bear small values below unity; typical experimental values, e.g., in rf discharge experiments, $\mu=7.14\times10^{-2}$ and $\nu=7.9\times10^{-2}$ (Ref. 11), are of similar order of magnitude to those reported by satellite observations: typically $T_h=10$ eV, $T_c=0.8$ eV, i.e., $\mu=8\times10^{-2}$ and $\nu=0.1$ (Ref. 15) or even possibly less.[10] However, higher experimental values, i.e., μ between 0.2 and 0.5 and ν varying from 0.33 to 6, have also been reported in discharge plasmas.[9] Both of the parameters ν and μ enter the model formulaes via the relations (7) [or (8)] and (9); to see this, notice that $N_c=n_{c,0}/(n_{c,0}+n_{h,0})=\nu/(1+\nu)$ and $T_{eff}=T_h(1+\nu)/(1+\nu/\mu)$, so that $\beta_c=(1+1/\nu)/(1+\nu/\mu)$.

The well-known IAW dispersion relation $\omega^2=c_{s,eff}^2 k^2/(k^2\lambda_{D,e}^2+1)$ (Ref. 27) is obtained from Eqs. (1)–(5). On the other hand, the system (6) yields the reduced relation $\omega^2=k^2/(k^2+1)$, which of course immediately recovers the former dispersion relation upon restoring dimensions.

B. Lagrangian formulation

Let us introduce the Lagrangian variables $\{\xi,\tau\}$, which are related to the Eulerian variables $\{x,t\}$ via the relations

$$\xi = x - \int_0^\tau u(\xi,\tau')d\tau', \quad \tau = t. \tag{10}$$

Notice that they coincide at $t=0$. The transformation (10) implies the gradient relations

$$\partial/\partial x \to \alpha^{-1}\,\partial/\partial\xi, \quad \partial/\partial t \to \partial/\partial\tau - \alpha^{-1}\,u\,\partial/\partial\xi,$$

where we have defined the quantity

$$\alpha(\xi,\tau) \equiv \frac{\partial x}{\partial \xi} = 1 + \int_0^\tau d\tau' \frac{\partial}{\partial\xi}u(\xi,\tau'). \tag{11}$$

Notice that the convective derivative $D \equiv \partial/\partial t + u\,\partial/\partial x$ is now plainly identified to $\partial/\partial\tau$. Also note that α satisfies $\alpha(\xi,\tau=0)=0$ and

$$\frac{\partial\,\alpha(\xi,\tau)}{\partial\,\tau} = \frac{\partial\,u(\xi,\tau)}{\partial\,\xi}. \tag{12}$$

As a matter of fact, the Lagrangian transformation defined here reduces to a Galilean transformation if one suppresses the evolution of u, i.e., for $u=\text{const}$ (viz, $\partial u/\partial\tau = \partial u/\partial\xi = 0$, hence $\alpha=1$). Furthermore, should one also suppress the dependence in time τ, it is reminiscent of the traveling wave ansatz $f(x,t)=f(x-vt\equiv s)$, which is widely used in the Sagdeev potential formalism (see, e.g., Refs. 4–7).

The variable transformation defined above leads to a new set of equations

$$n(\xi,\tau) = \alpha^{-1}(\xi,\tau)n(\xi,0), \tag{13}$$

$$\frac{\partial\,u(\xi,\tau)}{\partial\,\tau} = \frac{Z_i e}{m_i}E(\xi,\tau), \tag{14}$$

$$\alpha^{-1}(\xi,\tau)\frac{\partial\,E(\xi,\tau)}{\partial\,\xi} = 4\pi Z_i e[n(\xi,\tau) - \hat{n}n_{i,0}], \tag{15}$$

$$\left(\frac{\partial}{\partial\,\tau} - \alpha^{-1}u\frac{\partial}{\partial\,\xi}\right)E(\xi,\tau) = -4\pi\,Z_i e\;n(\xi,\tau)u(\xi,\tau), \tag{16}$$

where the electric field $\mathbf{E}$ is now related to the potential ϕ via

$$E(\xi,\tau) = -\alpha^{-1}(\xi,\tau)\frac{\partial\,\phi(\xi,\tau)}{\partial\,\xi}. \tag{17}$$

See that we have temporarily restored dimensions for physical transparency; recall that the (dimensionless) quantity $\hat{n}$, which is in fact a function of ϕ, is given by Eq. (7). One immediately recognizes the role of the (inverse of the) function $\alpha(\xi,\tau)$ as a density time evolution operator, cf. Eq. (13).[28] Poisson's equation is now obtained by combining Eqs. (15) and (17),

$$\alpha^{-1}\frac{\partial}{\partial\,\xi}\left(\alpha^{-1}\frac{\partial\,\phi}{\partial\,\xi}\right) = -4\pi Z_i e(n - \hat{n}n_{i,0}). \tag{18}$$

In principle, our aim is to solve the system of equations (13)–(16) or, by eliminating E, Eqs. (13), (14), and (18) for a given initial condition $n(\xi,\tau=0)=n_0(\xi)$, and then make use of the definition (10) in order to invert back to the Eulerian arguments of the state moment variables (i.e., density, velocity, etc.). However, this is admittedly not an easy task to accomplish.

III. NONLINEAR ION PLASMA OSCILLATIONS

Multiplying Eq. (15) by $u(\xi,\tau)$ and then adding to Eq. (16), one obtains

$$\frac{\partial\,E(\xi,\tau)}{\partial\,\tau} = -4\pi Z_i e n_{i,0}\hat{n}u(\xi,\tau). \tag{19}$$

Combining with Eq. (14), one obtains the equation

$$\frac{\partial^2 u}{\partial\,\tau^2} = -\omega_{p,i}^2\hat{n}u, \tag{20}$$

where $\omega_{p,i}$ is the ion plasma frequency (defined above). Despite its apparent simplicity, Eq. (20) is *neither* an ordinary differential equation—since all variables depend on *both* time τ and space ξ—*nor* a closed evolution equation for the mean velocity $u(\xi,\tau)$: note that the (normalized) background particle density $\hat{n}$ depends on the potential ϕ and on the plasma parameters; see its definition (7). The evolution of the potential $\phi(\xi,\tau)$, in turn, involves $u(\xi,\tau)$ [via the quantity $\alpha(\xi,\tau)$] and the ion density $n(\xi,\tau)$.

Equation (20) suggests that the system performs nonlinear oscillations at a frequency $\omega=\omega_{p,i}\hat{n}^{1/2}$. Near equilibrium, the quantity $\hat{n}$ is approximately equal to unity and one plainly recovers a linear oscillation at the ion plasma frequency $\omega_{p,i}$. Quite unfortunately this apparent simplicity, which might in principle enable one to solve for $u(\xi,\tau)$ and then obtain $\{\xi,\tau\}$ in terms of $\{x,t\}$ and vice versa (cf. Davidson's treatment for electron plasma oscillations in Ref. 24; also compare to Ref. 25, setting $\gamma=0$ therein), is absent in the general (off-equilibrium) case where the plasma oscillations described by Eq. (20) are intrinsically *nonlinear*.

Since Eq. (20) is in general not a closed equation for u, unless the background density $\hat{n}$ is constant (i.e., independent of ϕ, as in Refs. 24 and 25), one can neither apply standard methods involved in the description of nonlinear oscillators on Eq. (20) (cf. Ref. 25) nor reduce the description to a study of Eqs. (19) and (20) (cf. Ref. 23), but rather has to retain all (or rather five) of the evolution equations derived above, since five interdependent dynamical state variables (i.e., n, u, E, ϕ, and α) are involved. This procedure will be exposed in the following section.

IV. PERTURBATIVE NONLINEAR LAGRANGIAN TREATMENT

Let us consider weakly nonlinear oscillations performed by our system close to (but not at) equilibrium.

The basis of our perturbative study will be the reduced system of equations

$$\frac{\partial}{\partial\,\tau}(\alpha\;n) = 0,$$

$$\frac{\partial\,u}{\partial\,\tau} = E,$$

$$\frac{\partial E}{\partial \xi} = (n - \hat{n})\alpha,$$

$$\alpha E = - \frac{\partial \phi}{\partial \xi},$$

$$\frac{\partial \alpha}{\partial \tau} = \frac{\partial u}{\partial \xi}, \tag{21}$$

which follow from the Lagrangian equations (13)–(18) by scaling over appropriate quantities, as described in Sec. II A.[29] This system describes the evolution of the state vector $\mathbf{S}=(\alpha,n,u,E,\phi)$ $(\in \Re^5)$ in the Lagrangian coordinates defined above. We will consider small deviations from the equilibrium state $\mathbf{S}_0=(1,1,0,0,0)^T$, by taking $\mathbf{S}=\mathbf{S}^{(0)} + \epsilon \mathbf{S}_1{}^{(0)} + \epsilon^2 \mathbf{S}_2{}^{(0)} + \cdots$, where ϵ $(\ll 1)$ is a smallness parameter. Accordingly, we shall Taylor develop the quantity $\hat{n}(\phi)$ near $\phi \approx 0$, viz., $\phi \approx \epsilon\phi_1 + \epsilon^2\phi_2 + \cdots$, in order to express $\hat{n}$ as

$$\begin{aligned}\hat{n} &\approx 1 + c_1\phi + c_2\phi^2 + c_3\phi^3 + \cdots \\ &= 1 + \epsilon c_1\phi_1 + \epsilon^2(c_1\phi_2 + c_2\phi_1^2) \\ &\quad + \epsilon^3(c_1\phi_3 + 2c_2\phi_1\phi_2 + c_3\phi_1^3) + \cdots,\end{aligned} \tag{22}$$

where the coefficients c_j $(j=1,2,\ldots)$, which are determined from the definition (7) of $\hat{n}$, contain all the essential dependence on the plasma parameters $n_{c/h,0}$ and $T_{c/h}$ or, say, μ and ν; making use of $e^x \approx \sum_{n=0}^{\infty} x^n/n!$, one easily obtains

$$c_1 = Z_i^{-1}, \quad c_n = \frac{(n_c + n_h)(n_h T_c^n + n_c T_h^n)}{n!\, Z_i^n (n_h T_c + n_c T_h)^n} \quad (n \geq 2),$$

or, in terms of ν and μ,

$$c_n = \frac{(1+\nu)^{n-1}(\mu^n + \nu)}{n!\, Z_i^n (\mu + \nu)^n} \quad (n \geq 1).$$

Check that, for either $\nu=0$ or $\nu \to \infty$ (i.e., for vanishing minority electrons), one recovers the correct expression $c_n = 1/(n!\, Z_i^n)$.

According to the standard reductive perturbation technique,[30] we shall consider the stretched (slow) Lagrangian coordinates $Z=\epsilon(\xi - \lambda\tau)$, $T=\epsilon^2\tau$ (where $\lambda \in \Re$ will be determined later). The perturbed state of (the jth—$j=1,\ldots,5$—component of) the state vector $\mathbf{S}^{(n)}$ is assumed to depend on the fast scales via the carrier phase $\theta = k\xi - \omega\tau$, while the slow scales enter the argument of the (jth element's) lth harmonic amplitude $S_{j,l}^{(n)}$, viz., $Sj^{(n)} = \sum_{l=-\infty}^{\infty} S_{j,l}^{(n)} \times (Z,T) e^{il(k\xi - \omega\tau)}$ (where $S_{j,-l}^{(n)} = S_{j,l}^{(n)*}$ ensures reality). Treating the derivative operators as

$$\frac{\partial}{\partial \tau} \to \frac{\partial}{\partial \tau} - \epsilon\lambda \frac{\partial}{\partial Z} + \epsilon^2 \frac{\partial}{\partial T}, \quad \frac{\partial}{\partial \xi} \to \frac{\partial}{\partial \xi} + \epsilon \frac{\partial}{\partial Z},$$

and substituting into the system of evolution equations, one obtains an infinite series in both (perturbation order) ϵ^n and (phase harmonic) l. The standard perturbation procedure now consists in solving in successive orders $\sim \epsilon^n$ and substituting in subsequent orders. The details of the method, involving a calculation which is particularly lengthy yet perfectly straightforward, is outlined, e.g., in Ref. 27, so only the essential stepstones are provided here.

The equations obtained for $n=l=1$ determine the first harmonics of the perturbation

$$n_1^{(1)} = -\alpha_1^{(1)} = (k^2/\omega^2)\psi,$$

$$u_1^{(1)} = (k/\omega)\psi, \quad E_1^{(1)} = -ik\psi \tag{23}$$

where ψ denotes the potential correction $\phi_1^{(1)}$. The cyclic frequency ω obeys the dispersion relation $\omega^2 = k^2/(k^2 + sc_1)$, which exactly recovers, once dimensions are restored, the standard IAW dispersion relation[1] mentioned above.

Proceeding in the same manner, we obtain the second-order quantities, namely, the amplitudes of the second harmonics $\mathbf{S}_2^{(2)}$ and constant ("direct current") terms $\mathbf{S}_0^{(2)}$, as well as a finite contribution $\mathbf{S}_1^{(2)}$ to the first harmonics; as expected from similar studies, these three (sets of 5, at each n,l) quantities are found to be proportional to ψ^2, $|\psi|^2$, and $\partial\psi/\partial Z$, respectively; the lengthy expressions are omitted here for brevity. The $(n=2,\ l=1)$ equations provide the compatibility condition: $\lambda = \omega(1-\omega^2)/k = d\omega/dk$; λ is therefore the group velocity $v_g(k) = \omega'(k)$ at which the wave envelope propagates. It turns out that v_g decreases with increasing wave number k; nevertheless, it always remains positive.

In order $\sim \epsilon^3$, the equations for $l=1$ yield an explicit compatibility condition in the form of a nonlinear Schrödinger-type equation (NLSE)

$$i\frac{\partial \psi}{\partial T} + P\frac{\partial^2 \psi}{\partial Z^2} + Q|\psi|^2\psi = 0. \tag{24}$$

Recall that $\psi \equiv \phi_1^{(1)}$ denotes the (envelope) amplitude of the first-order electric potential perturbation. The "slow" variables $\{Z,T\}$ were defined above.

The *dispersion coefficient* P is related to the curvature of the dispersion curve as $P = \omega''(k)/2 = -3\omega^3(1-\omega^2)/(2k^2)$. One may easily check that P is negative (for all values of k). Note that P depends on neither μ nor ν.

The *nonlinearity coefficient* Q is due to carrier wave self-interaction. It is given by the expression

$$\begin{aligned}Q = -\frac{\omega^3}{12k^4}\big[&(3c_1^2 - 2c_2)^2 + 3(5c_1^3 + 4c_1c_2 - 6c_3)k^2 \\ &+ 3(c_1^2 + 8c_2)k^4 - 3c_1k^6\big],\end{aligned} \tag{25}$$

where the coefficients $c_{1,2,3}$ were defined above. Let us investigate the behavior of Q in some limiting cases. First, for low wave number k, Q goes to $-\infty$ as

$$Q \approx -\frac{[\mu^2(\nu-2) + \nu - 6\mu\nu - 2\nu^2]^2}{12(\mu+\nu)^4}\frac{1}{k}.$$

In the single electron population limit $\nu \to 0$ (i.e., $n_e \ll n_h$), one obtains the approximate expression

$$Q_{\nu \to 0} \approx \frac{3k^6 - 15k^4 - 18k^2 - 4}{12k(1+k^2)^{3/2}}$$

(see that the μ dependence also disappears, naturally). Finally, in the vanishing cold electron temperature limit $\mu \to 0$ (viz., $T_e \ll T_h$), one has

$$Q_{\mu\to 0} \approx [3\nu^2k^6 - 3\nu(4+5\nu)k^4 + (3-18\nu^2)k^2 - (1-2\nu)^2]/[12\nu^2 k(1+k^2)^{3/2}]. \tag{26}$$

It may be noted, for rigor, that the dependence of ψ on $Z=\epsilon(\xi-v_g\tau)$, which arises as a compatibility condition (see above), here essentially amounts to $\psi=\psi(Z_1)$, where $Z_1=\epsilon(x-v_g t)+O(\epsilon^2)$ [to see this, combine the definitions (10) with the assumption $u=\epsilon u^{(1)}+\epsilon^2 u^{(2)}+\cdots$]; recall that Eq. (24) is valid at order ϵ^3 (since $\partial/\partial T, \partial^2/\partial Z^2 \sim \epsilon^2$, $\psi\sim\epsilon$), i.e., within an error $\sim\epsilon^4$ being neglected.

A. Stability analysis

Following the standard stability analysis,[27,31–33] we may linearize around the plane wave solution of the NLSE (24) $\psi=\hat{\psi}e^{iQ|\hat{\psi}|^2\tau}$+c.c. (c.c.—complex conjugate)—notice the amplitude dependence of the frequency shift $\Delta\omega=\epsilon^2 Q|\hat{\psi}|^2$—by setting $\hat{\psi}=\hat{\psi}_0+\epsilon\hat{\psi}_1$, and then assuming the perturbation $\hat{\psi}_1$ to be of the form $\hat{\psi}_1=\hat{\psi}_{1,0}\, e^{i(\hat{k}\zeta-\hat{\omega}\tau)}$+c.c.. Substituting into Eq. (24), one thus readily obtains $\hat{\omega}^2=P^2\hat{k}^2[\hat{k}^2-2(Q/P)|\hat{\psi}_{1,0}|^2]$. The wave will thus be *stable* ($\forall\ \hat{k}$) if the product PQ is negative. However, for positive $PQ>0$, instability sets in for wave numbers below a critical value $\hat{k}_{cr}=\sqrt{2Q/P}|\hat{\psi}_{1,0}|$, i.e., for wavelengths above a threshold $\lambda_{cr}=2\pi/\hat{k}_{cr}$; defining the instability growth rate $\sigma=|\mathrm{Im}\hat{\omega}(\hat{k})|$, we see that it reaches its maximum value for $\hat{k}=\hat{k}_{cr}/\sqrt{2}$, viz.,

$$\sigma_{max} = |\mathrm{Im}\hat{\omega}|_{\hat{k}=\hat{k}_{cr}/\sqrt{2}} = |Q||\hat{\psi}_{1,0}|^2.$$

We see that the instability condition depends only on the sign of the product PQ, which may be studied numerically, relying on the expressions derived above. As a first result, we see that Q is always negative for low k, therefore prescribing instability for large wavelengths λ. Admittedly, this result qualitatively somehow contradicts the Eulerian behavior (for low k) predicted in Ref. 19; however, this is rather not unexpected, since we have relied on a nonlinear variable transformation, which undoubtedly modifies the nonlinear laws governing the system's evolution.

B. Finite amplitude nonlinear excitations

The NLSE (24) is long known to possess distinct types of localized constant profile (solitary wave) solutions, depending on the sign of the product PQ.[19,31–34] Remember that this equation here describes the evolution of the wave's envelope, so these solutions represent slowly varying localized envelope structures, confining the (fast) carrier wave. Following Ref. 34, we may seek a solution of Eq. (24) in the form $\psi(\zeta,\tau)=\rho(Z,T)e^{i\Theta(\zeta,\tau)}$+c.c., where ρ, σ are real variables which are determined by substituting into the NLSE and separating real and imaginary parts. The different types of solution thus obtained are summarized in the following.

For $PQ>0$ we find the (*bright*) *envelope soliton*

$$\rho = \pm\rho_0\, \mathrm{sech}\left(\frac{Z-u_e\tau}{L}\right),$$

$$\Theta = \frac{1}{2P}\left[u_e Z - \left(\Omega+\frac{1}{2}u_e^2\right)T\right], \tag{27}$$

which represents a localized pulse traveling at the envelope speed u_e and oscillating at a frequency Ω (at rest). The pulse width L depends on the maximum amplitude square ρ_0 as $L=(2P/Q)^{1/2}/\rho_0$. Since the product PQ is always positive for long wavelengths, as we saw above, this type of excitation will be rather privileged in real plasmas where a second population of (cold) electrons is present. The bright-type envelope soliton is depicted in Fig. 1; notice the resemblance to envelope electrostatic structures reported during satellite observations.[20,21]

For $PQ<0$ we obtain the *dark* envelope soliton (*hole*)[34]

$$\rho = \pm\rho_1\left[1-\mathrm{sech}^2\left(\frac{Z-u_eT}{L'}\right)\right]^{1/2} = \pm\rho_1 \tanh\left(\frac{Z-u_eT}{L'}\right),$$

$$\Theta = \frac{1}{2P}\left[u_e Z - \left(\frac{1}{2}u_e^2 - 2PQ\rho_1\right)T\right], \tag{28}$$

which represents a localized region of negative wave density (shock) traveling at a speed u_e. Again, the pulse width depends on the maximum amplitude square ρ_1 via $L'=(2|P/Q|)^{1/2}/\rho_1$.

Finally, still for $PQ<0$, one also obtains the *gray* envelope solitary wave[34]

$$\rho = \pm\rho_2\left[1-a^2\,\mathrm{sech}^2\left(\frac{Z-u_eT}{L''}\right)\right]^{1/2}, \tag{29}$$

which also represents a localized region of negative wave density. Comparing to the dark soliton (28), we note that the maximum amplitude ρ_2 is now finite (nonzero) everywhere; also, the pulse width of this gray-type excitation $L''=\sqrt{2|P/Q|}/(a\,\rho_2)$ now also depends on an independent parameter a which represents the modulation depth $(0<a\leqslant 1)$. The lengthy expressions which determine the phase shift Θ and the parameter a, which are omitted here for brevity, can be found in Refs. 19 and 34. For $a=1$, one recovers the dark soliton presented above.

An important qualitative result to be retained is that the envelope soliton width L and maximum amplitude ρ satisfy $L\rho\sim\sqrt{P/Q}$ (see above), and thus depend on (the ratio of) the coefficients P and Q; for instance, regions with higher values of P (or lower values of Q) will support wider (spatially more extended) localized excitations, for a given value of the maximum amplitude. Contrary to the KdV soliton picture, the width of these excitations does not depend on their velocity. It does, however, depend on the parameters μ and ν.

The localized excitations presented above represent the slowly varying envelope which confines the (fast) carrier space and time oscillations, viz., $\phi=\Psi(X,Z)\cos(k\xi-\omega\tau)$ for the electric potential ϕ [and analogous expressions for the density n_i, etc.; cf. Eq. (23)]. The different types of these envelope excitations are depicted in Figs. 1 and 2. Note the qualitative similarity between Fig. 1 and envelope modulated

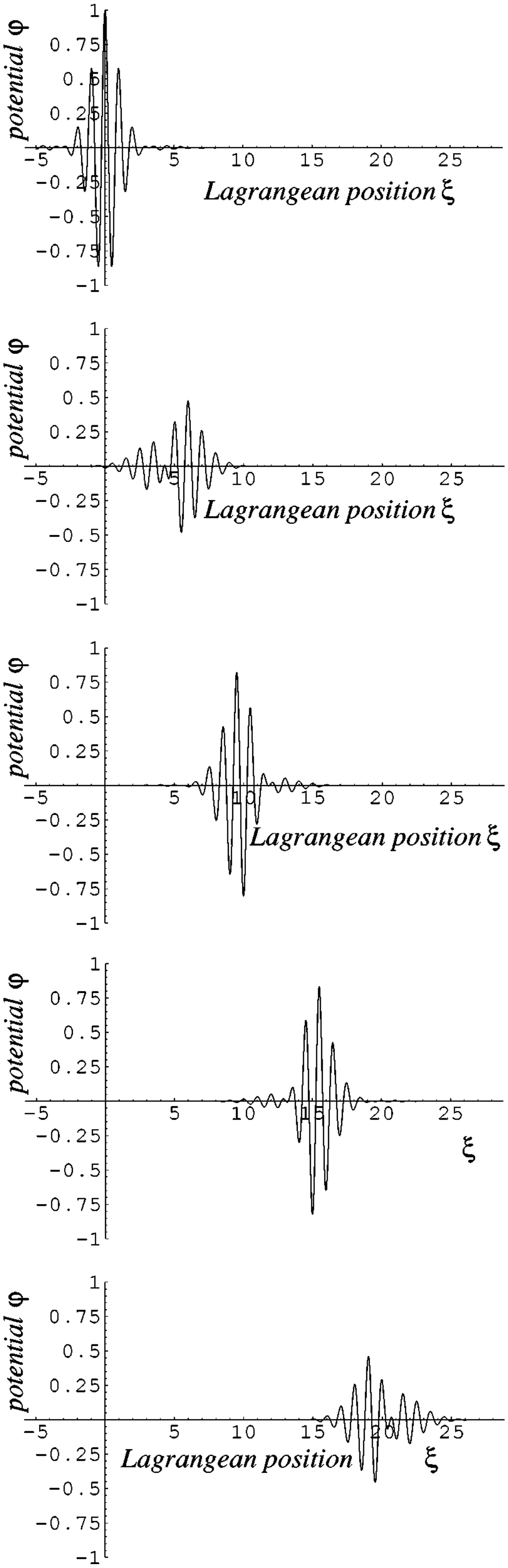

FIG. 1. The evolution of a modulated wave packet confined by a bright-type (pulse) envelope solution of the NLS equation, at different snapshots at $t_1 < t_2 < \cdots < t_5$; these wave forms have been computed directly from Eq. (27), only varying t, for the same (arbitrary) parameter values.

electrostatic wave forms reported by satellite observations; see, e.g., Fig. 1 in Ref. 20 or Fig. 4 in Ref. 21.

The qualitative characteristics of envelope excitations, which may possibly exist in the system under consideration, may be predicted by studying the ratio $P/Q \equiv \eta(k;\nu,\mu)$: recall that its sign determines the type (bright or dark) of the excitation, while its (absolute) value determines its width for a given amplitude (and vice versa). In Figs. 3–5 we have attempted to trace the complex behavior of η as a function of the wave number k and the parameters ν and μ. For a given wave number and temperature ratio $\mu = T_c/T_h$, varying the density ratio $\nu = n_c/n_h$ (i.e., injecting cold electrons) may result in a change in the form of the envelope excitations; see

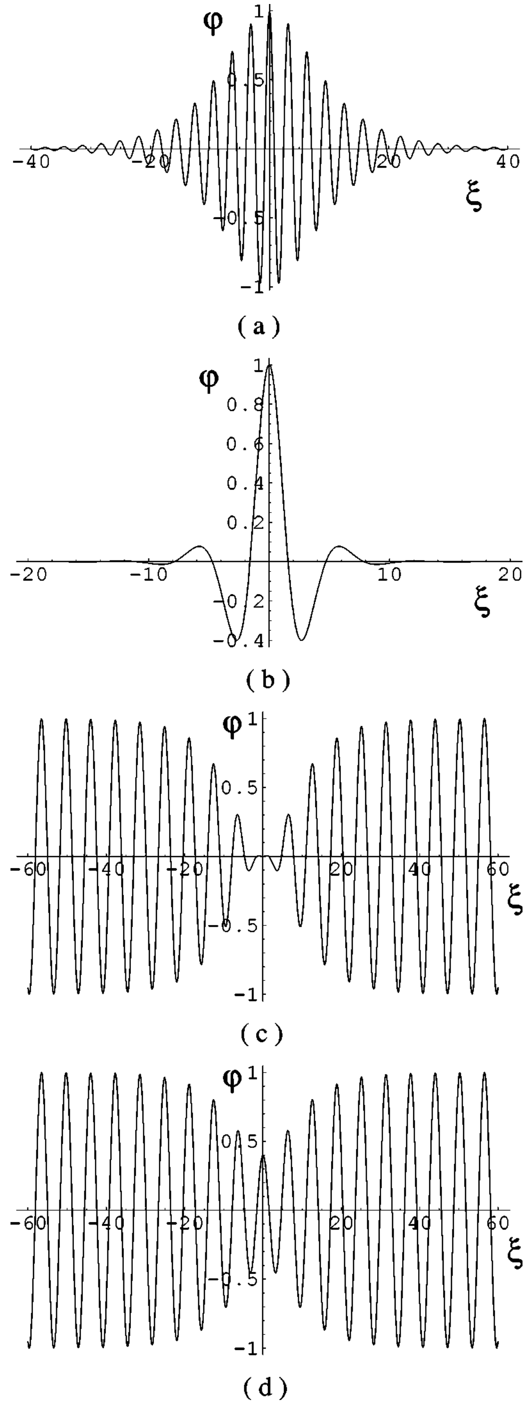

FIG. 2. A heuristic representation of different wave packets modulated by a soliton solution of the NLS equation. These excitations are of the (a,b) bright type ($PQ > 0$, pulses), (c) dark type, (d) gray type ($PQ < 0$, voids). Notice that the amplitude never reaches zero in (d).

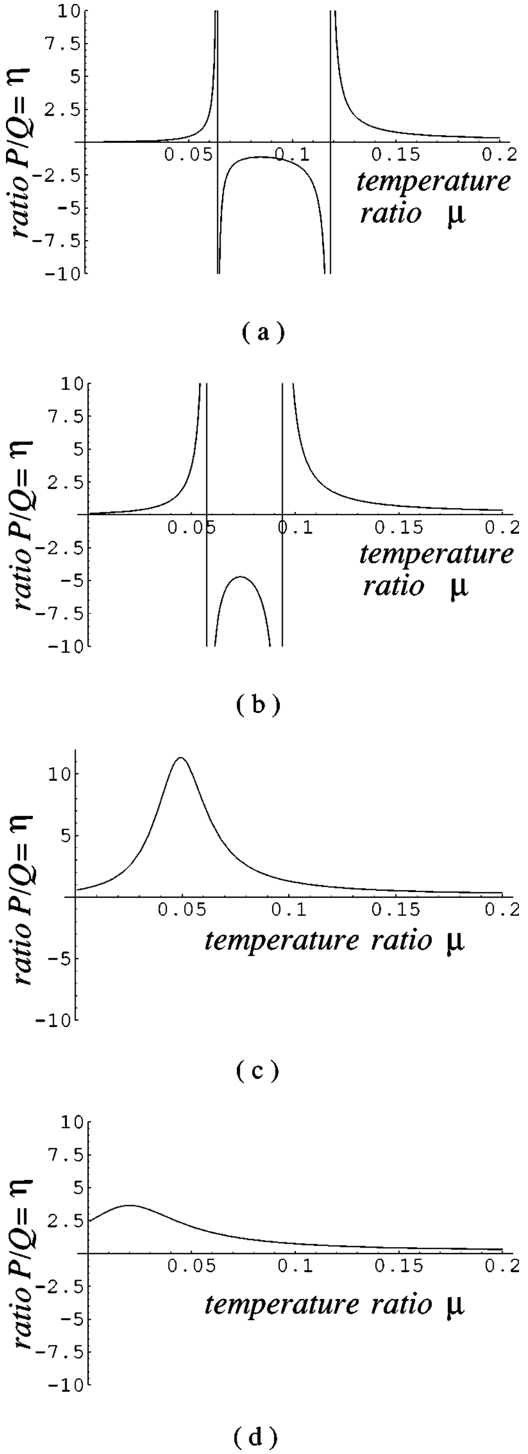

FIG. 3. The ratio $\eta = P/Q$ is depicted vs the temperature ratio $\mu = T_c/T_h$ for a wave number $k = 0.2k_{D,h}$ and a density ratio $\nu = n_c/n_h$ equal to (a) 0.1; (b) 0.2; (c) 0.3; (d) 0.4.

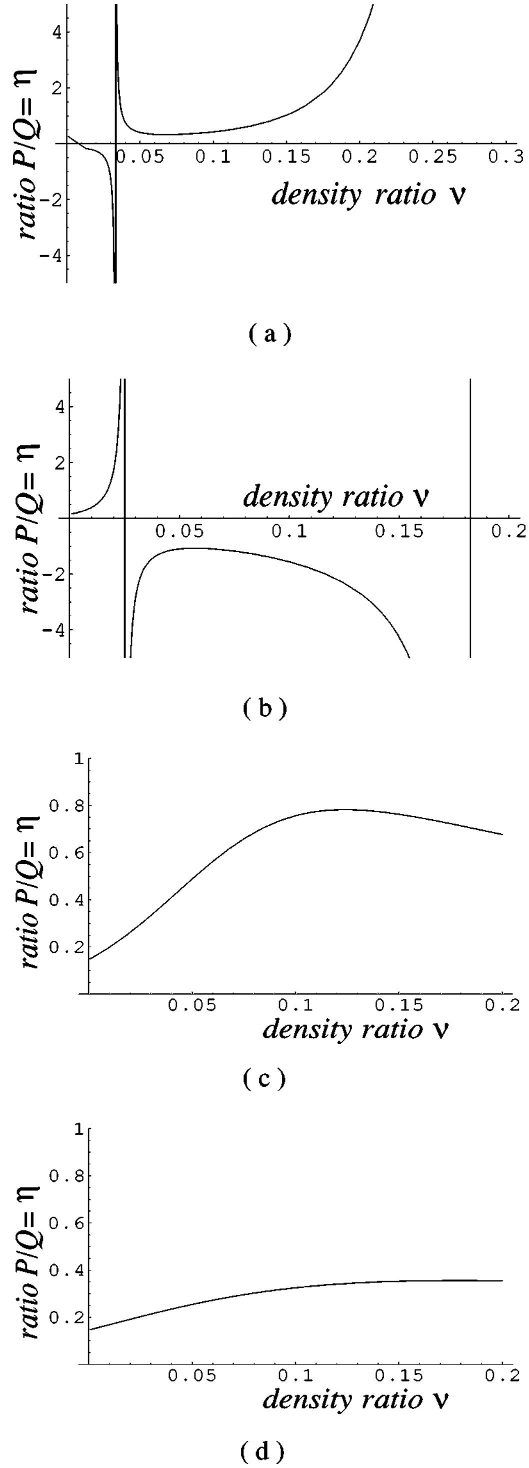

FIG. 4. The ratio $\eta = P/Q$ is depicted vs the density ratio $\nu = n_c/n_h$ for a wave number $k = 0.2k_{D,h}$ and a temperature ratio $\mu = T_c/T_h$ equal to (a) 0.05; (b) 0.1; (c) 0.15; (d) 0.2.

Fig. 3. The same is true when one modifies the temperature of the cold electron component, by keeping its density fixed; see Fig. 4. In the same time, different values of the carrier wave number may correspond to different excitations, even for a fixed pair of values of ν and μ. Since, intuitively speaking, different ion-acoustic wave numbers may be excited, for any given plasma conditions, none of the above soliton types is *a priori* excluded at some given situation. Furthermore, a simultaneous propagation of nonlinear excitations (moving at different velocities and not interacting with each other) is in principle possible, which leads to the conjecture that a local envelope soliton superposition may account for the apparent asymmetries observed in the potential and density variation structures reported by satellite observations. In particular, these results are in qualitative agreement with conclusions in Ref. 4, where it was argued that both compressive

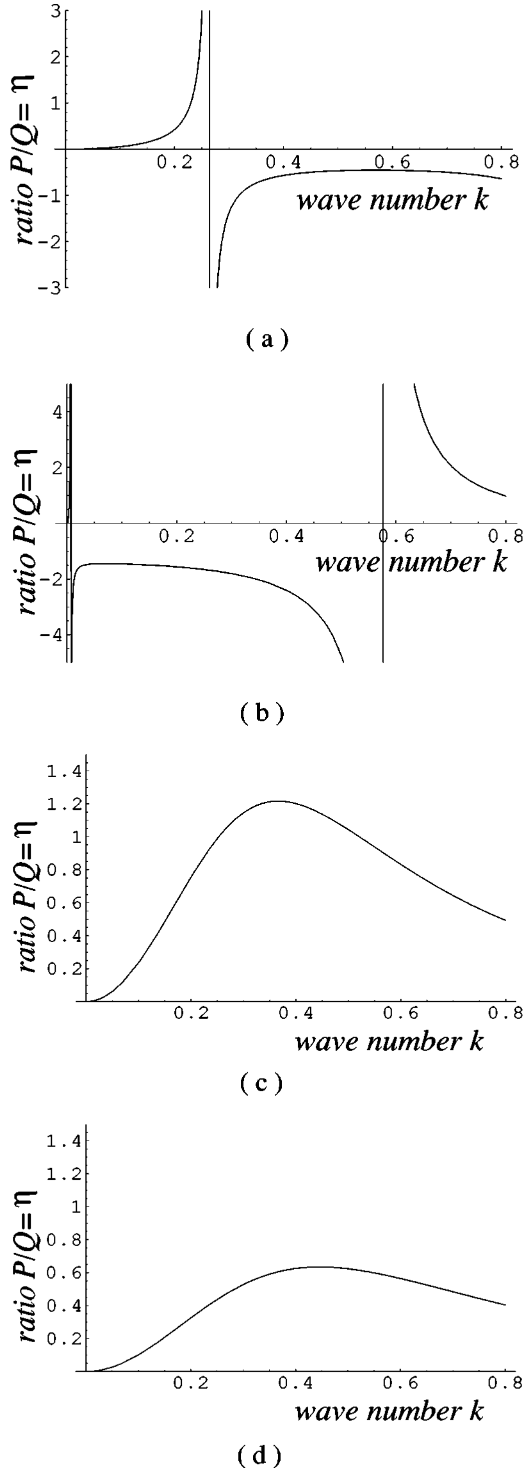

FIG. 5. The ratio $\eta = P/Q$ is depicted vs the wave number k for a density ratio $\nu = n_c/n_h = 0.1$ and a temperature ratio $\mu = T_c/T_h$ equal to (a) 0.05; (b) 0.1; (c) 0.15; (d) 0.2.

and rarefactive solitary structures may coexist, in the presence of an important temperature difference.

V. DISCUSSION AND CONCLUSIONS

We have studied the nonlinear propagation of ion-acoustic waves in a plasma which is characterized by two distinct electron temperatures. This study was motivated by satellite observations of electrostatic waves in the magnetosphere, where such a coexistence between electrons at different temperatures occurs. Employing a Lagrangian formalism, which extends known previous results on electron plasma oscillations, we have investigated the modulational stability of the amplitude of the propagating ion-acoustic oscillations and have shown that these electrostatic waves may become unstable, due to self-interaction of the carrier wave. This instability may either lead to wave collapse or to wave energy localization, in the form of propagating localized envelope structures. We have provided a exact set of analytical expressions for these localized excitations, which bear qualitative characteristics close to those of the modulated envelope electrostatic structures recently observed. Hopefully, these results may help us to explain the appearance of envelope localized wave forms in the Earth's auroral acceleration region and the magnetosphere, which—contrary to localized pulses/shocks which are effectively described via KdV or Sagdeev-potential theories—still lack a satisfying theoretical explanation.

This study complements a similar investigation which was recently carried out, based on an Eulerian formulation of the plasma fluid model.[19] As a matter of fact, our results provide a slightly different nonlinear stability profile for the wave amplitude, with respect to the previous (Eulerian) description; this was intuitively expected, since the passing to Lagrangian variables involves an inherently nonlinear transformation, which inevitably modifies the nonlinear evolution profile of the system described. However, the general qualitative result remains intact: the ion-acoustic-type electrostatic plasma waves may propagate in the form of localized envelope excitations, which are formed as a result of the mutual balance between dispersion and nonlinearity in the plasma fluid. More sophisticated descriptions, incorporating, e.g., thermal or collisional effects, may be elaborated in order to refine the parameter range of the problem, and will be reported later.

ACKNOWLEDGMENTS

This work was supported by the European Commission (Brussels) through the Human Potential Research and Training Network via the project entitled: "Complex Plasmas: The Science of Laboratory Colloidal Plasmas and Mesospheric Charged Aerosols" (Contract No. HPRN-CT-2000-00140). Support from the Deutsche Forschungsgemeinschaft through the SFB 591 German Government Program (*Sonderforschungsbereich 591 – Universelles Verhaltengleichgewichtsferner Plasmen: Heizung, Transport und Strukturbildung*) is also gratefully acknowledged.

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[28] Rigorously speaking, Eq. (13) was obtained from the (Lagrange transformed) density equation, which is reduced to $\partial(n\alpha)/\partial\tau=0$ by using the property (12); Eq. (13) then simply follows.
[29] In specific, Eqs. (21) are derived from Eqs. (13)–(15), $E=-\nabla\phi$, and Eq. (12), respectively. We have chosen to avoid the appearance of α^{-1}—cf. Eqs. (16) and (18)—for analytical convenience, and also to keep E in the description, for physical transparency.
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