

## Weakly nonlinear vertical dust grain oscillations in dusty plasma crystals in the presence of a magnetic field

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The weakly nonlinear regime of transverse paramagnetic dust grain oscillations in dusty (complex) plasma crystals is discussed. The nonlinearity, which is related to the sheath electric/magnetic field(s) and to the intergrain (electrostatic/magnetic dipole) interactions, is shown to lead to the generation of phase harmonics and, in the case of propagating transverse dust-lattice modes, to the modulational instability of the carrier wave due to self-interaction. The stability profile depends explicitly on the form of the electric and magnetic fields in the plasma sheath. The long term evolution of the modulated wave packet, which is described by a nonlinear Schrödinger-type equation, may lead to propagating localized envelope structures whose exact forms are presented and discussed. Explicit suggestions for experimental investigations are put forward. © 2004 American Institute of Physics. [DOI: 10.1063/1.1763577]

### I. INTRODUCTION

Dusty (or complex) plasmas (DP) have been attracting an increasing interest among plasma physicists for more than a decade due to the appearance of many novel phenomena. Of particular importance is the possibility of the existence of new plasma configurations (states), due to the strong intergrain coupling, including the spontaneous formation of crystal-like DP structures, when charged microparticles (dust grains) are trapped in the sheath region between the electrodes, in plasma discharge experiments.<sup>1,2</sup> Such crystals, which are generally horizontally arranged in a position of equilibrium which is levitated above the negative electrode in discharge experiments, have been shown to support longitudinal (acoustic) as well as transverse (optical) oscillation modes.<sup>1</sup>

Force equilibrium in DP crystals is “traditionally” ensured by the sheath electric force which exactly balances gravity at the levitation height. Crystalline complex plasma structures have been observed in recent rf discharge experiments<sup>3</sup> in which the plasma sheath was embedded in an external magnetic field. Gravity compensation was thus attributed to magnetic forces, thanks to the paramagnetic properties of the magnetized dust grains. Theoretical studies then followed for the investigation of conditions for magnetic-field-assisted crystal equilibria involving paramagnetic charged dust grains. The role of various forces acting on paramagnetic grains has been discussed in Ref. 4, where magnetic forces (due to magnetic dipole interactions) have been shown to prevail over the (weaker) electric polarization forces (especially under experimental conditions considered

in Ref. 3, where magnetic fields as strong as a few thousand Gauss were used). According to these considerations, the properties of transverse magnetized dust lattice (TMDL) oscillations were investigated,<sup>5</sup> where both isolated grain linear oscillatory modes and propagating linear waves were discussed. The dependence of the dynamics of (linear) grain oscillations on the specific characteristics of the (inhomogeneous) magnetic field was studied in detail in Ref. 5, and the possible occurrence of a linear instability depending on the field profile was pointed out.

It is now established that the investigation of the linear regime of a dynamical system only unveils part of its dynamical profile. Nonlinearity may be present in DP crystal dynamics, either due to intergrain interactions or due to the sheath environment. In a generic manner, as the oscillation amplitude becomes large, nonlinearity is first manifested via the generation of wave harmonics leading to the amplitude modulation of the waves. Increasing displacements even further, one may come up with the formation of localized structures (solitons), which propagate and interact with each other in a remarkably stable manner, thanks to a compensation between nonlinearity and dispersion. The present study is devoted to an investigation of weakly nonlinear effects, with respect to magnetized transverse dust-lattice oscillations. Nonlinearity is explicitly shown to be related to the magnetic and/or electric field profile, as well as to the intergrain electrostatic and/or magnetic (dipole-dipole) interactions. By using a perturbative approach, the generation of phase harmonics is elucidated and exact expressions are obtained for the (weak) vertical displacement of the paramagnetic dust particles. Once these oscillations propagate in the dusty crystal as a transverse wave, the amplitude is shown to be potentially unstable to external perturbations, under conditions which depend explicitly on the magnetic field characteristics. Finally, the possible formation of localized envelope excitations is discussed, and the dependence of these coherent

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structure characteristics on the plasma parameters is pointed out.

## II. NONLINEAR SINGLE GRAIN OSCILLATIONS

Let us consider the vertical motion of a charged dust grain (mass  $M$  and charge  $q$ ), subject to an external static electric and magnetic field  $\mathbf{E}$  and  $\mathbf{B}$ , respectively, both in the vertical ( $\sim \hat{z}$ ) direction. The vertical displacement  $\delta z = z - z_0$  from the equilibrium position  $z_0$  obeys the equation of motion

$$\ddot{z} = \frac{F}{M} - g - \nu \dot{z}, \tag{1}$$

where we have set  $z_0 = 0$ . The three terms in the right-hand side (rhs) of Eq. (1) account for the (sum of the) electric and magnetic forces, viz.  $F = F_e + F_m$ , in the  $z$  direction, the force of gravity, and the usual (Epstein) damping term, which involves the phenomenological damping rate  $\nu$  due to dust-neutral collisions.

We shall assume a smooth, continuous variation of the (generally inhomogeneous) field intensities  $\mathbf{E}$  and  $\mathbf{B}$ , as well as the grain charge  $q$  (which may vary due to charging processes<sup>6</sup>) near the equilibrium position  $z_0 = 0$ . Thus, we may develop

$$E(z) \approx E_0 + E'_0 z + \frac{1}{2} E''_0 z^2 + \dots,$$

$$B(z) \approx B_0 + B'_0 z + \frac{1}{2} B''_0 z^2 + \dots,$$

and

$$q(z) \approx q_0 + q'_0 z + \frac{1}{2} q''_0 z^2 + \dots,$$

where the prime denotes differentiation with respect to  $z$  and the subscript “0” denotes evaluation at  $z = z_0$ , viz.,  $E_0 = E(z = z_0)$ ,  $E'_0 = dE(z)/dz|_{z=z_0}$ , and so forth. Accordingly, the electric force  $F_e = q(z)E(z)$  and the magnetic force  $F_m = -\partial(mB)/\partial z = -2\alpha B \partial B/\partial z$  [where the grain magnetic moment  $\mu$  is related to the grain radius  $a$  and permeability  $\mu$  via  $m = (\mu - 1)a^3 B/(\mu + 2) \equiv \alpha B$  (Ref. 7)], which are now expressed as

$$F_e(z) \approx q_0 E_0 + (q_0 E'_0 + q'_0 E_0) z + \frac{1}{2} (q_0 E''_0 + 2q'_0 E'_0 + q''_0 E_0) z^2 + \dots$$

and

$$F_m(z) \approx -2\alpha B_0 B'_0 - 2\alpha (B_0 B''_0 + B'_0 B''_0) z - \alpha (B_0 B'''_0 + 3B'_0 B''_0) z^2 + \dots,$$

may be combined into

$$F_e + F_m = -\frac{\partial \Phi}{\partial z},$$

where we have introduced the phenomenological potential  $\Phi(z)$

$$\begin{aligned} \Phi(z) \approx & \Phi(z_0) + \frac{\partial \Phi}{\partial z} \Big|_{z=z_0} z + \frac{1}{2!} \frac{\partial^2 \Phi}{\partial z^2} \Big|_{z=z_0} z^2 \\ & + \frac{1}{3!} \frac{\partial^3 \Phi}{\partial z^3} \Big|_{z=z_0} z^3 + \dots \equiv \Phi_0 + \Phi_{(1)} z + \frac{1}{2} \Phi_{(2)} z^2 \\ & + \frac{1}{6} \Phi_{(3)} z^3 + \dots \end{aligned} \tag{2}$$

The definitions of  $\Phi_{(j)} \equiv (\partial^j \Phi(z)/\partial z^j)|_{z=z_0} = -(qE_0)_0^{(j-1)} + \alpha(B^2)_0^{(j)}$  [for  $j = 1, 2, \dots$ ; the superscript  $(j)$  denotes order in partial differentiation] are obvious

$$\begin{aligned} \Phi_{(1)} &= -(qE)_0 + \alpha(B^2)'_0 = -q_0 E_0 + 2\alpha B_0 B'_0 \\ \Phi_{(2)} &= -(qE_0)'_0 + \alpha(B^2)''_0 \\ &= -(q'_0 E_0 + q_0 E'_0) + 2\alpha(B_0 B''_0 + B'_0 B''_0) \\ \Phi_{(3)} &= -(qE_0)''_0 + \alpha(B^2)'''_0 \\ &= -(q''_0 E_0 + 2q'_0 E'_0 + q_0 E''_0) \\ &\quad + 2\alpha(3B'_0 B''_0 + B_0 B'''_0), \end{aligned} \tag{3}$$

and so forth. Note the force balance equation

$$Mg = q_0 E_0 - 2\alpha B_0 B'_0,$$

which is satisfied at equilibrium.

Given the above definitions, the equation of motion (1) takes the form

$$\ddot{z} + \nu \dot{z} + \omega_g^2 z + K_1 z^2 + K_2 z^3 = 0, \tag{4}$$

where

$$\omega_g^2 = \Phi_{(2)}/M, \quad K_1 = \Phi_{(3)}/(2M), \quad K_2 = \Phi_{(4)}/(6M), \tag{5}$$

and higher order terms are omitted. One immediately notices the intrinsically nonlinear character of the transverse dust oscillations due to the electric/magnetic field inhomogeneity. In the linear limit ( $\Phi_{(j)} = 0$  for  $j \geq 3$ ), the results of Ref. 5 are exactly recovered.

Once the set of parameter values are determined, Eq. (4) can be solved, e.g., via a Lindstedt–Poincaré method. Assuming small displacements of the form  $u(t) = \sum_{n=1}^{\infty} \epsilon^n u_n(\tau)$ , where the reduced time variable  $\tau$  incorporates a frequency shift due to nonlinearity:  $\tau = \omega t = (1 + \epsilon \lambda_1 + \epsilon^2 \lambda_2 + \dots) \omega_g t \equiv (1 + \lambda) \omega_g t$ . One thus obtains the solution

$$u(\tau) \approx \epsilon A \cos \tau + \epsilon^2 \left( \frac{K_1}{3\omega_g^2} A^2 \cos 2\tau - \frac{2K_1}{\omega_g^2} |A|^2 \right) + \dots, \tag{6}$$

along with a nonlinear frequency shift  $\delta\omega = \lambda \omega_g \sim |A|^2 \omega_g$ . We see that the nonlinearity of the force acting on the paramagnetic dust grains typically results in the generation of frequency harmonics once the oscillation amplitude becomes slightly significant in magnitude. The occurrence of this effect depends on the relative magnitude of the coefficients  $K_1$  and  $K_2$  in Eq. (4). This phenomenon has already been studied for an unmagnetized dust crystal (viz., for  $B = 0$ )

(Ref. 8); based on Eq. (9) therein, one deduces that  $\Phi_{(3)}/\Phi_{(2)} = -1 \text{ mm}^{-1}$  and  $\Phi_{(4)}/\Phi_{(2)} = +0.42 \text{ mm}^{-2}$ , implying  $K_1/\omega_g^2 = -0.5 \text{ mm}^{-1}$  and  $K_2/\omega_g^2 = +0.07 \text{ mm}^{-2}$ . Similar data on the magnetized DP crystal case, which are not yet available, may be deduced from appropriate experiments. This type of analysis may be pursued further once such feedback from experiments is available.

### III. NONLINEAR TRANSVERSE MAGNETIZED DUST-LATTICE OSCILLATIONS

Let us now consider a layer of identical charged dust grains (of lattice constant  $r_0$ ). The Hamiltonian of such a chain is of the form

$$H = \sum_n \frac{1}{2} M \left( \frac{d\mathbf{r}_n}{dt} \right)^2 + \sum_{m \neq n} U(r_{nm}) + \Phi_{ext}(\mathbf{r}_n),$$

where  $\mathbf{r}_n$  is the position vector of the  $n$ th grain.  $U_{nm}(r_{nm})$  is a binary interaction potential function, related to the electrostatic intergrain interaction potential  $\phi(x)$  (typically of the Debye type, though ion flow in the sheath may be included for a more complete description,<sup>9</sup> as well as the magnetic moment of the  $n$ th and  $m$ th grains, which are located at a distance  $r_{nm} = |\mathbf{r}_n - \mathbf{r}_m|$ . Even though the analytical form of  $U(r)$  need not be specified here, for generality, we may explicitly refer to the model of Refs. 3, 5 in the case of magnetized dust crystals [see Eq. (15) in Ref. 5, or Eq. (8) in Ref. 3]. The external potential  $\Phi_{ext}(\mathbf{r})$  accounts for the external force fields in the sheath region [i.e., essentially  $\Phi$  as defined in Eq. (2) above]; nevertheless, in a more sophisticated description,  $\Phi_{ext}$  may include the parabolic horizontal confinement potential imposed in experiments for stability,<sup>10</sup> or the initial laser excitation triggering the oscillations in experiments (both neglected here).

Considering the motion of the  $n$ th dust grain in the *transverse* (vertical, off-plane,  $\sim \hat{z}$ ) direction, we have the equation of motion including dissipation caused by dust-neutral collisions

$$M \left( \frac{d^2 z_n}{dt^2} + \nu \frac{dz_n}{dt} \right) = - \sum_n \frac{\partial U_{nm}(r_{nm})}{\partial z_n} + F_{ext}(z_n), \quad (7)$$

where  $F_{ext} = -\partial\Phi_{ext}(z)/\partial z$  accounts for all external forces in the  $z$  direction.

#### Equation of motion

Assuming small displacements from equilibrium, one may Taylor expand the interaction potential  $U(r)$  around the equilibrium intergrain distance  $lr_0 = |n - m|r_0$  (between  $l$ th order neighbors,  $l = 1, 2, \dots$ ), i.e., around  $z_n \approx 0$ . Retaining only nearest-neighbor interactions ( $l = 1$ ), we then obtain from Eq. (6)

$$\begin{aligned} \frac{d^2 z_n}{dt^2} + \nu \frac{dz_n}{dt} + \omega_g^2 z_n + K_1 z_n^2 + K_2 z_n^3 \\ = \omega_{0,T}^2 (2z_n - z_{n+1} - z_{n-1}) \\ + K_3 [(z_{n+1} - z_n)^3 - (z_n - z_{n-1})^3], \end{aligned} \quad (8)$$

The transverse oscillation characteristic frequency  $\omega_{0,T}$  and the coefficient  $K_3$  are defined by

$$\begin{aligned} \omega_{0,T} &= [-U'(r_0)/(Mr_0)]^{1/2}, \\ K_3 &= -\frac{1}{2Mr_0^3} [U'(r_0) - r_0 U''(r_0)]. \end{aligned} \quad (9)$$

The gap frequency  $\omega_g$  and the coefficients  $K_1$  and  $K_2$  have been defined above. Of course, a negative/positive value of  $U'(r_0)/U''(r_0)$  is a condition ensuring stability of transverse dust-lattice (TDL) oscillations, i.e., a real-valued frequency  $\omega_{0,T}/\omega_g$  ( $\cdot/\dagger$  here means  $\cdot$  or  $\dagger$ , respectively); cf. Eq. (9) above.

Note that the equation of motion presented in Ref. 5 [see Eq. (18) therein] is exactly recovered in the linear case (i.e., for  $K_1 = K_2 = K_3 = 0$ ), given the form of the interaction potential in that model. In particular, the frequency  $\omega_{0,T}$  then becomes<sup>5</sup>

$$\omega_{0,T}^2 = \frac{q^2}{Mr_0^3} \left( 1 + \frac{r_0}{\lambda_D} \right) \exp\left(-\frac{r_0}{\lambda_D}\right) + 9 \frac{m_0^2}{r_0^5},$$

where  $\lambda_D$  denotes the Debye length; one immediately distinguishes the contribution from the Debye potential (first term) from the second term, which is due to the magnetic moment  $m_0 = \alpha B_0$  (see the definitions above). Note, in passing, that the vertical motion equation of the recent nonlinear model by Ivlev *et al.*<sup>11</sup> is exactly recovered in the appropriate limit.<sup>12</sup>

In general (i.e., regardless of the particular analytical aspects of each model), one immediately notes that nonlinearity may either arise from the sheath environment (electric/magnetic fields) or from the interactions between the paramagnetic dust grains.

Adopting the standard continuum approximation, often employed in solid state physics,<sup>13</sup> we may assume that only small displacement variations occur between neighboring sites, viz.,

$$z_{n\pm 1} = u \pm r_0 \frac{\partial u}{\partial x} + \frac{1}{2} r_0^2 \frac{\partial^2 u}{\partial x^2} \pm \frac{1}{3!} r_0^3 \frac{\partial^3 u}{\partial x^3} + \frac{1}{4!} r_0^4 \frac{\partial^4 u}{\partial x^4} \pm \dots,$$

where the vertical displacement  $z_n(t)$  is now expressed by a continuous function  $u = u(x, t)$ . One may now proceed by inserting this ansatz in the discrete equation of motion (8), and carefully evaluate each term. The calculation leads to a continuum equation of motion of the form

$$\begin{aligned} \ddot{u} + \nu \dot{u} + c_0^2 u_{xx} + c_0^2 \frac{r_0^2}{12} u_{xxx} = -\omega_g^2 u - K_1 u^2 - K_2 u^3 \\ + K_3 r_0^4 (u_x^3)_x, \end{aligned} \quad (10)$$

where terms of higher nonlinearity have been omitted. We have defined the characteristic TDLW velocity  $c_0 = \omega_{0,T} r_0$ ; the subscript  $x$  denotes the partial differentiation, viz.,  $(u_x^3)_x = 3(u_x)^2 u_{xx}$ .

The continuum equation of motion (10) is a modified, damped Boussinesq-like nonlinear equation. In the following, we omit the phenomenological damping term (which may be added *a posteriori*, once an explicit solution for the displacement is obtained).

**IV. AMPLITUDE MODULATION—A NONLINEAR SCHRÖDINGER EQUATION**

According to the standard reductive perturbation method,<sup>14</sup> we shall consider a small displacement of the form:  $u = \epsilon u_1 + \epsilon^2 u_2 + \dots$ , where  $\epsilon \ll 1$  is a small parameter and solutions  $u(x, t)$  at each order are assumed to be a sum of  $m$ th order harmonics, viz.  $u_n = \sum_{m=0}^n u_m^{(n)} \exp[i(kx - \omega t)]$  (the reality condition  $u_{-m}^{(n)} = u_m^{(n)*}$  is understood). Time and space scales are accordingly expanded as

$$\begin{aligned} \partial/\partial t &\rightarrow \partial/\partial T_0 + \epsilon \partial/\partial T_1 + \epsilon^2 \partial/\partial T_2 + \dots \\ &\equiv \partial_0 + \epsilon \partial_1 + \epsilon^2 \partial_2 + \dots \end{aligned}$$

and

$$\begin{aligned} \partial/\partial x &\rightarrow \partial/\partial X_0 + \epsilon \partial/\partial X_1 + \epsilon^2 \partial/\partial X_2 + \dots \\ &\equiv \nabla_0 + \epsilon \nabla_1 + \epsilon^2 \nabla_2 + \dots, \end{aligned}$$

implying that

$$\begin{aligned} \partial^2/\partial t^2 &\rightarrow \partial_0^2 + 2\epsilon \partial_0 \partial_1 + \epsilon^2 (\partial_1^2 + 2\partial_0 \partial_2) + \dots, \\ \partial^2/\partial x^2 &\rightarrow \nabla_0^2 + 2\epsilon \nabla_0 \nabla_1 + \epsilon^2 (\nabla_1^2 + 2\nabla_0 \nabla_2) + \dots \end{aligned}$$

and

$$\partial^4/\partial x^4 \rightarrow \nabla_0^4 + 4\epsilon \nabla_0^3 \nabla_1 + 2\epsilon^2 \nabla_0^2 (3\nabla_1^2 + 2\nabla_0 \nabla_2) + \dots.$$

This reductive perturbation technique is a standard procedure, often used in the study of the nonlinear wave propagation (e.g., in hydrodynamics, in nonlinear optics, etc.).<sup>15</sup>

We now proceed by substituting all the above series in Eq. (10) and isolating terms arising in the equation of motion at each order in  $\epsilon^n$ . By solving for  $u_n$ , then substituting in the following order  $\epsilon^{n+1}$  and so forth, we obtain the  $m$ th harmonic amplitudes  $u_n^{(m)}$  at each order, along with a compatibility condition up to any given order.

The equation obtained in order  $\sim \epsilon^1$  is

$$\left( \partial_0^2 - c_0^2 \nabla_0^2 - c_0^2 \frac{r_0^2}{12} \nabla_0^4 \right) u_1 \equiv L_0 u_1 = 0. \tag{11}$$

We may assume that  $u_1 = u_1^{(1)} \exp[i(kx - \omega t)] + \text{c.c.} \equiv \exp(i\theta) + \text{c.c.}$ , where  $\omega$ ,  $k = 2\pi/\lambda$  and  $\lambda$  denote the carrier wave frequency, wave number, and wavelength, respectively; c.c. stands for the complex conjugate. The dispersion relation obtained from Eq. (11) is of the form

$$\omega^2 = \omega_g^2 - k^2 c_0^2 + k^4 \frac{c_0^2}{12} r_0^2 = \omega_g^2 - \omega_{T,0}^2 k^2 r_0^2 \left( 1 - \frac{1}{12} k^2 r_0^2 \right), \tag{12}$$

predicting an inverse optical-mode-like behavior (in agreement with previous results; in fact, this is exactly the continuum analog of the discrete dispersion relation in Ref. 5). We assume that the reality condition  $\omega > 0$  is, in principle, satisfied in the region of validity of the continuum hypothesis  $k \ll (\pi/r_0) \equiv k_{cr,0}$  (nevertheless, note that the possibility of magnetic field-related instabilities, basically related to the relative magnitude of  $\omega_g^2$  and  $\omega_{T,0}^2$ , was put forward in Ref. 5).

Let us evaluate the action of the linear operator  $L_0$ , defined above, on higher harmonics of the phase  $\theta$

$$\begin{aligned} L_0 e^{in\theta} &= [(-in\omega)^2 + \omega_g^2 + c_0^2 (ink)^2 \\ &\quad + c_0^2 (r_0^2/12)(ink)^4] e^{in\theta} \\ &= [n^2(n^2 - 1)c_0^2 (r_0^2/12)k^4 - (n^2 - 1)\omega_g^2] e^{in\theta} \\ &\equiv D_n e^{in\theta}, \end{aligned} \tag{13}$$

where we made use of the dispersion relation (12); thus  $D_0 = \omega_g^2$ ,  $D_1 = 0$ , and  $D_2 = -3\omega_g^2 + c_0^2 r_0^2 k^4$ .

To order  $\sim \epsilon^2$ , the condition of suppression of secular terms takes the form

$$\frac{\partial u_1^{(1)}}{\partial T_1} + v_g \frac{\partial u_1^{(1)}}{\partial X_1} = 0, \tag{14}$$

i.e.,  $u_1^{(1)} = u_1^{(1)}(\zeta)$ , where  $\zeta = \epsilon(x - v_g t)$ , implying that the slowly varying wave envelope travels at the *negative group velocity*  $v_g = \omega'(k) = -(1 - k^2 r_0^2/6)c_0^2 k/\omega$ ; this *backward wave* has been observed experimentally.<sup>16</sup> We note that  $v_g$  becomes zero at  $k = \sqrt{6} r_0^{-1}$ , beyond which it becomes positive; nevertheless, note for rigor that this comment rather becomes obsolete once one properly takes into account the influence of the lattice discreteness on the form of the dispersion relation  $\omega = \omega(k)$ ; cf. Ref. 17.

This procedure finally leads to a solution of the form

$$\begin{aligned} u(x, t) &= \epsilon [A e^{i(kx - \omega t)} + \text{c.c.}] + \epsilon^2 K_1 \left[ -\frac{2|A|^2}{\omega_g^2} \right. \\ &\quad \left. + \frac{A^2}{-D_2} e^{2i(kx - \omega t)} + \text{c.c.} \right] + \mathcal{O}(\epsilon^3), \end{aligned} \tag{15}$$

where  $\omega$  obeys the dispersion relation (12).

The amplitude  $A(X, T)$  obeys a *nonlinear Schrödinger equation* (NLSE) of the form

$$i \frac{\partial A}{\partial T} + P \frac{\partial^2 A}{\partial X^2} + Q |A|^2 A = 0, \tag{16}$$

where the “slow” variables  $\{X, T\}$  are  $\{X_1 - v_g T_1, T_2\}$ , respectively. The *dispersion coefficient*  $P$ , which is related to the curvature of the phonon dispersion relation (12) as  $P = (1/2)(d^2\omega/dk^2)$ , reads

$$P = -\frac{c_0^2 \omega_g^2}{2\omega^3} \left( 1 - \frac{1}{2} k^2 r_0^2 \right), \tag{17}$$

and the *nonlinearity coefficient*

$$Q = \frac{1}{2\omega} \left[ 2K_1^2 \left( \frac{2}{\omega_g^2} + \frac{1}{D_2} \right) - 3K_2 - 3K_3 k^4 r_0^4 \right] \tag{18}$$

is related to both the sheath electric/magnetic field and the intergrain coupling nonlinearity discussed above. As expected, Eq. (18) recovers exactly the previously derived expressions (in similar studies) in the appropriate limits, i.e., precisely Eq. (9) in Ref. 18 in the dispersionless limit [for very low  $k$ , i.e., canceling terms in  $k^4$  in (12)], and Eq. (10) in Ref. 17 (where a quasicontinuum approximation was adopted, i.e., a continuum envelope but discrete carrier wave description). Note however that intergrain interaction nonlinearity was neglected in those studies (i.e.,  $K_3 = 0$  therein).

In view of the analysis that follows, it may be appropriate to cast Eq. (16) in a nondimensional form, for physical clarity. This is readily done by introducing a set of appropri-

ately chosen scales, e.g., the lattice constant  $r_0$  and the inverse eigenfrequency  $\omega_g^{-1}$ , i.e., by scaling the (slow) variables  $X$  and  $T$  as  $X \rightarrow X' = X/r_0$  and  $T \rightarrow T' = \omega_g T$ ; the vertical displacement amplitude becomes  $A \rightarrow A' = A/r_0$ . The form of Eq. (16) is then exactly recovered upon substituting with  $P \rightarrow P' = P/(\omega_g r_0^2)$  and  $Q \rightarrow Q' = Q r_0^2/\omega_g$ . The primes will be dropped in the following.

**A. Modulational instability**

In a generic manner, a modulated wave whose amplitude obeys the NLS equation (16) is stable/unstable to perturbations if the product  $PQ$  is negative/positive. To see this, one may first check that the NLSE is satisfied by the monochromatic solution (Stokes' wave)  $A(X, T) = A_0 e^{iQ|A_0|^2 T} + c.c.$  The standard (linear) stability analysis then shows that a linear modulation with the frequency  $\Omega$  and the wave number  $\kappa$  obeys the dispersion relation

$$\Omega^2(\kappa) = P^2 \kappa^2 \left( \kappa^2 - 2 \frac{Q}{P} |A_0|^2 \right), \tag{19}$$

which exhibits a purely growing mode if  $\kappa \leq \kappa_{cr} = (Q/P)^{1/2} |A_0|$ . The growth rate attains a maximum value  $\gamma_{max} = Q|A_0|^2$ . This mechanism is known as the *Benjamin-Feir instability*.<sup>15</sup> For  $PQ < 0$ , the wave is modulationally stable, as evident from Eq. (19).

Note that the coefficient  $P$  is negative, given the quasi-inverse-parabolic form of the dispersion curve  $\omega(k)$  for low  $k$ . One therefore only needs to deduce the sign of  $Q$ , given by Eq. (18), in order to determine the stability profile of the TDL oscillations. In fact, given the definitions (5), (9) of the parameters  $\omega_g$ ,  $K_1$ ,  $K_2$ , and  $K_3$ , one may easily derive an expression for  $Q$  in terms of the (derivatives of the) potentials  $\Phi(z)$  (for the sheath) and  $U(z)$  (for inter-grain interactions). The exact form of the potential  $\Phi(z)$  (hence the coefficients  $\Phi_{(j)}$ ,  $j = 1, 2, \dots$ ) may be obtained from *ab initio* calculations or by experimental data fitting. As mentioned above, some evidence for the numerical values of the coefficients  $\Phi_{(j)}$ , yet only in the absence of the magnetic field, can be found in Ref. 8, where the dust grain potential energy was reconstructed from experimental data [see Eq. (9) therein]; those values (cf. comment above) seem to suggest that the value of  $Q$  for low wave numbers  $k$  is positive, as may be readily checked from Eq. (18). Therefore, under the experimental conditions described in Ref. 8, the TDL wave would propagate as a *stable* wave, for large wavelength values  $\lambda$ . However, for shorter wavelengths, either the coefficient  $P = \omega''(k)/2$  or  $Q$  may change sign, and the TDL wave may thus be potentially unstable. These results should *a priori* be checked by appropriately designed experiments, in particular with regard to magnetically levitated DP crystals.

**B. Localized excitations**

A final comment concerns the possibility of the existence of localized excitations related to TMDL waves. It is known that the NLSE (16) admits localized solutions (envelope solitons) of the *bright* ( $PQ > 0$ ) or *dark/gray* ( $PQ < 0$ ) type. These expressions are found by inserting the trial function  $A = A_0 \exp(i\Theta)$  in Eq. (16) and then separating real and

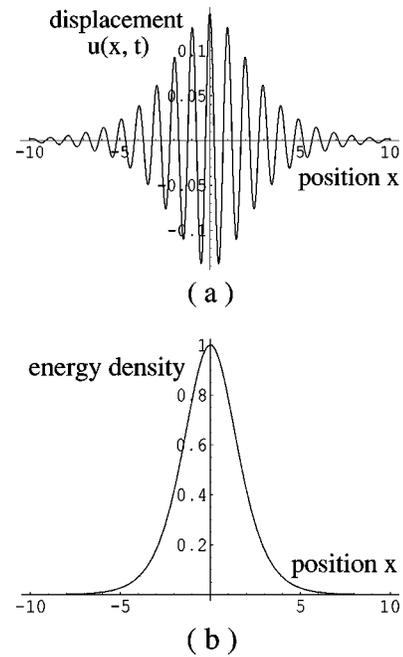


FIG. 1. *Bright* type (pulse) soliton solution ( $PQ > 0$ ) of the NLS equation (16), obtained for an indicative set of numerical values for the parameters in Eqs. (20), (21):  $\epsilon = 0.1$ ,  $P = Q = 1$ ,  $v_e = 0.2$  (i.e.,  $\rho = 2A_0 = \sqrt{2}$ ),  $k = 2\pi/\lambda = 2\pi$ , and  $L = 2$ : (a) the wave form, as results from Eq. (20); (b) the corresponding (localized) wave energy ( $\sim \rho^2$ )—normalized value.

imaginary parts in order to determine the (real) functions  $A_0(X, T)$  and  $\Theta(X, T)$ . Details on the derivation of their analytic form can be found, e.g., in Refs. 19,20, so only the final expressions will be given in the following. Let us retain that this *ansatz* amounts to a total (vertical) grain displacement  $u(x, t)$  essentially equal to

$$u(x, t) = \epsilon \rho \cos(kx - \omega t + \Theta), \tag{20}$$

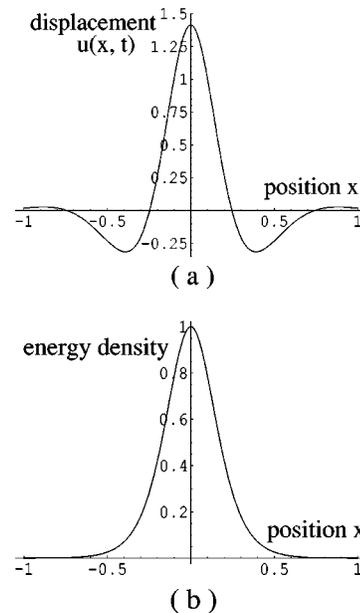


FIG. 2. Same label and data as in Fig. 1, except  $L = 0.2$ .

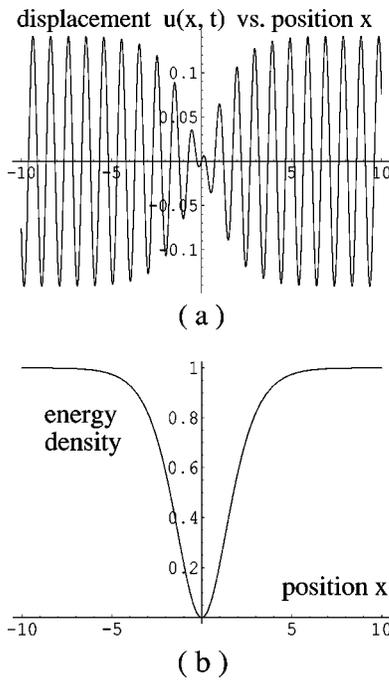


FIG. 3. *Dark* type soliton solutions of the NLS equation for  $PQ < 0$  (holes); here  $Q = -P = 1$  and the remaining values are just as in Fig. 1: (a) the wave form, as results from Eq. (20); (b) the corresponding (localized) wave energy ( $\sim \rho^2$ ), normalized over its asymptotic value.

where  $\rho = 2A_0$  and the nonlinear phase shift is  $\Theta \sim \epsilon$ , since linear in  $\{X, T\} = \{\epsilon x, \epsilon t\}$  (see below).

The *bright* type (*pulse*) envelope solutions (*continuum breathers*, see Figs. 1, 2), obtained for  $PQ > 0$ , are given by

$$A = \left(\frac{2P}{QL^2}\right)^{1/2} \operatorname{sech}\left(\frac{X - v_e T}{L}\right) \times \exp\left\{i \frac{1}{2P} \left[ v_e X + \left( \Omega - \frac{v_e^2}{2} \right) T \right]\right\}, \quad (21)$$

where  $v_e$  is the envelope velocity;  $L$  and  $\Omega$  represent the pulse's spatial width and oscillation frequency, respectively. In our problem, the bright-type localized envelope solutions may occur and propagate in the lattice if a sufficiently short wavelength is chosen, so that the product  $PQ$  is positive. We note that the pulse width  $L$  and the amplitude  $A_0$  satisfy  $LA_0 = (2P/Q)^{1/2} = \text{const}$ . Let us point out that, when the pulse's spatial width  $L$  is comparable in order of magnitude to the carrier wavelength  $\lambda$ , this (now highly localized) type of solution is similar in structure to the (odd-parity) discrete breather (DB) modes (or intrinsic localized modes, ILMs) recently widely studied in molecular chains.<sup>21</sup>

For  $PQ < 0$ , we have the *dark* envelope soliton (*hole*)<sup>19</sup> (see Fig. 3)

$$A = \pm A_0 \tanh\left(\frac{X - v_e T}{L'}\right) \times \exp\left\{i \frac{1}{2P} \left[ v_e X + \left( 2PQA_0^2 - \frac{v_e^2}{2} \right) T \right]\right\}, \quad (22)$$

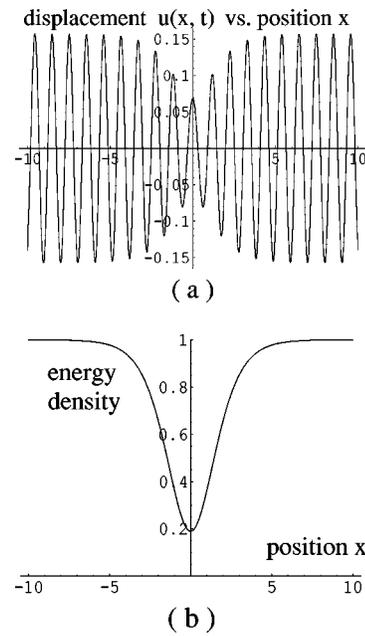


FIG. 4. *Gray* type soliton solutions of the NLS equation for  $PQ < 0$ : values identical to those in Fig. 3, in addition to  $V_0 = 0.5$ ,  $d = 0.9$ . Note that the amplitude never reaches zero.

which represents a localized region of negative wave density (void). The pulse width  $L' = (2|P/Q|)^{1/2}/A_0$  is inversely proportional to the amplitude  $A_0$ .

For  $PQ < 0$ , we also have the *gray* envelope soliton<sup>19</sup> (see Fig. 4)

$$A = A_0 \{1 - d^2 \operatorname{sech}^2\{[X - v_e T]/L''\}\}^{1/2} \exp(i\Theta), \quad (23)$$

where  $\Theta = \Theta(X, T)$  is a nonlinear phase correction to be determined by substituting into the NLSE (16); the calculation yields the complex expression:

$$\Theta = \frac{1}{2P} \left[ V_0 X - \left( \frac{1}{2} V_0^2 - 2PQA_0^2 \right) T + \Theta_0 \right] - S \sin^{-1} \frac{d \tanh\left(\frac{X - v_e T}{L''}\right)}{\left[ 1 - d^2 \operatorname{sech}^2\left(\frac{X - v_e T}{L''}\right) \right]^{1/2}} \quad (24)$$

[see Eq. (64) in Ref. 19a]. This localized excitation also represents a localized region of negative wave density,  $\Theta_0$  is a constant phase,  $S$  denotes the product  $S = \operatorname{sgn}(P) \times \operatorname{sgn}(v_e - V_0)$ . Again, the pulse width  $L' = (|P/Q|)^{1/2}/(dA_0)$  is inversely proportional to the amplitude  $A_0$ , and now also depends on an independent real parameter  $d$ , which regulates the modulation depth;  $d$  is given by:  $d^2 = 1 + (v_e - V_0)^2 / (2PQA_0^2) \leq 1$ .  $V_0$  is an independent real constant which satisfies (see details in Ref. 19):  $V_0 - \sqrt{2|PQ|A_0^2} \leq v_e \leq V_0 + \sqrt{2|PQ|A_0^2}$ . This excitation represents a localized region of negative wave density (a *void*), with finite amplitude at  $X = 0$ . For  $d = 1$  (thus  $V_0 = v_e$ ), one recovers the *dark* envelope soliton presented above, which is characterized by a vanishing amplitude at  $X = 0$ .

We should admit, for rigor, that the latter excitations (of dark/gray type), yet apparently privileged in the continuum

limit (where  $PQ < 0$  for low  $k$ ), are rather physically irrelevant in our (infinite chain) model, since they correspond to an infinite energy stored in the lattice. Nevertheless, their existence locally in a finite-sized chain may be considered (and possibly confirmed) either numerically or experimentally.

It may be stressed that the grain displacement corresponding to the (envelope) excitations presented here is intrinsically different in form (and obeys different physics) from the pulslike small-amplitude localized structures (solitons) typically found via Korteweg–de Vries theories; see Refs. 19, 20 for a critical comparison.

## V. CONCLUSIONS

The present study has been devoted to an investigation of amplitude modulation effects associated with either isolated transverse dust-grain oscillations or propagating transverse dust-lattice waves in DP crystals of paramagnetic charged dust particles, embedded in an external magnetic field. We have shown that nonlinearity comes into play once the oscillation amplitude slightly departs from the weak-displacement (linear) regime. This nonlinearity, which is related to the electric and/magnetic field(s) in the sheath region, affects the dynamics of transverse dust-lattice oscillations via the generation of phase harmonics and the potential instability of the carrier wave, due to self-interaction. The latter may presumably be responsible for energy localization in the DP crystal via the formation of localized envelope excitations. Analytic expressions for these excitations are presented and briefly discussed. It should be stressed that the instability suggested here, related to self-modulation of the carrier wave and triggered once the amplitude becomes slightly important, is completely independent from the one pointed out in Ref. 5, which is related to the values of the intrinsic parameters (the gap frequency, in particular) involved in the physics of the problem.

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