# Nonlinear modulation of transverse dust lattice waves in complex plasma crystals

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The occurrence of the modulational instability in transverse dust lattice waves propagating in a one-dimensional dusty plasma crystal is investigated. The amplitude modulation mechanism, which is related to the intrinsic nonlinearity of the sheath electric field, is shown to destabilize the carrier wave under certain conditions, possibly leading to the formation of localized envelope excitations. Explicit expressions for the instability growth rate and threshold are presented and discussed.
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Studies of various collective processes[1] in dust contaminated plasmas (DP) have been of significant interest in connection with linear and nonlinear waves that are observed in laboratory and space plasmas. An issue of particular importance is the formation of strongly coupled DP crystals by highly charged dust grains, for instance in the sheath region above a horizontal negatively biased electrode in experiments.[2–5] Low-frequency oscillations occurring in these mesoscopic dust grain quasi-lattices, in both longitudinal and transverse directions, have been theoretically predicted,[6–11] and later experimentally observed.[1,11–15] We note that the observation of the characteristics of transverse vibrations around a levitated equilibrium position, where the electric and gravity forces are in balance, has been suggested as a diagnostic tool, enabling the determination of the grain charge.[4,16,17] Recent generalizations taking into account dust charge variations,[18] layer coupling[19] and two-dimensional crystal anisotropy[20] are also worth mentioning.

It is known from solid-state physics[21] that lattice vibrations are inevitably subject to amplitude modulation due to intrinsic nonlinearities of the medium. Furthermore, the wave propagation in crystals are often characterized by the Benjamin–Feir-type modulational instability (MI), a well-known mechanism for the energy localization related to the wave propagation in nonlinear dispersive media. The MI mechanism has been thoroughly studied in the past, mostly in one-dimensional (1D) solid state systems, where nonlinearities of the substrate potential and/or particle coupling may be seen to destabilize waves and possibly lead to localized excitations (solitary waves).[22,23] In the context of plasma wave theory, this nonlinear mechanism has been investigated in a variety of contexts since long ago.[24,25] In a weakly coupled dusty (or complex) plasma (DP), in particular, new electrostatic wave modes arise,[1,26] whose modulation has been studied quite recently;[27,28] instability conditions were shown to depend strongly on modulation obliqueness, dust concentration and the ion temperature.[28]

In principle, nonlinearity is always present in dusty plasma vibrations, due to the form of the inter-grain electrostatic interaction potential, which may be of the Debye–Hückel-type[6,9] or else.[29,30] Furthermore, the electric potential dominating oscillations in the transverse direction, yet often thought to be practically parabolic near the levitated equilibrium position,[31] is intrinsically anharmonic,[32] as suggested by experimental results.[13,33] Despite this evidence, knowledge of nonlinear mechanisms related to low-frequency DP lattice modes is still in a preliminary stage. Small amplitude localized longitudinal excitations (described by a Korteweg–de Vries equation) were considered in Refs. 6 and 34 based on which longitudinal dust lattice wave (LDLW) amplitude modulation was considered in Ref. 35. However, to the best of our knowledge, no study has been carried out, from first principles, of the amplitude modulation of transverse dust lattice waves (TDLWs) because of the sheath electric field nonlinearity. This Letter aims in making a first analytical step towards the study of DP crystals in this framework.

About three years ago, Misawa *et al.*[14] reported the observation of vertical nonlinear oscillations of a dust grain in a plasma sheath, and interpreted their results in terms of a position-dependent delayed grain charging effect. Zafiu *et al.*[33] studied vertical dust-grain oscillations in a rf-discharge plasma, and succeeded in pointing out their strongly nonlinear behavior due to the sheath potential anharmonicity. Recently, Ivlev *et al.*[36] investigated the nonlinear coupling between high-frequency transverse (vertical) dust lattice oscillations (TDLOs) and slow longitudinal dust lattice vibrations (LDLVs). Within the framework of a slowly varying envelope approximation, they derived a pair of equations for the modulated TDLOs and driven (by the ponderomotive force of the latter) slow LDLVs. The coupled system of equations admit envelope soliton solutions. Finally, compressional pulses were studied, both theoretically and experi-

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mentally, in Ref. 37; however, these excitations are related to longitudinal grain motion.

In this Brief Communication, we consider the amplitude modulation of transverse dust lattice waves (TDLWs), taking into account self-interaction nonlinearities associated with harmonic generation, involving the intrinsic nonlinearity of the sheath electric potential. Thus, the underlying physics of our nonlinear process is entirely different from those considered in Refs. 14, 33, 36, and 37. By adopting the reductive perturbation technique, we derive a cubic nonlinear Schrödinger equation for the modulated TDLWs. It is shown that the latter may be modulationally unstable depending on the plasma parameters, and that they can propagate in the form of envelope localized excitations due to a balance between nonlinearity and dispersion. Explicit forms of localized excitations are presented.

Let us consider TDLWs (vertical, off-plane) propagating in the one-dimensional (1D) DP crystal. DP crystals have been shown to support low-frequency optical-mode-like oscillations in both transverse and longitudinal directions.[1,6–10] Focusing on the former and summarizing previous results, let us recall that transverse motion of a charged dust grain (mass $M$, charge $q$, both assumed constant for simplicity) in a DP crystal (lattice constant $r_0$) obeys an equation of the form

$$M\frac{d^2\delta z_n}{dt^2}=M\omega_0^2(2\delta z_n-\delta z_{n-1}-\delta z_{n+1})+F_e-Mg, \quad (1)$$

where $\delta z_n=z_n-z_0$ denotes the small displacement of the $n$th grain around the equilibrium position $z_0$, in the transverse $(z-)$ direction, propagating in the longitudinal $(x-)$ direction. Assuming that the neighboring dust grains (situated at a distance $x=|x_i-x_j|$) interact via an electrostatic potential $\Phi(x)$, we obtain the DP oscillation "eigenfrequency" $\omega_0^2=-(q/Mr_0)((\partial\Phi(x)/\partial x)|_{x=r_0}$, e.g., in the case of a Debye–Hückel potential: $\Phi(x)=(q/x)e^{-x/\lambda_D}$,

$$\omega_0^2=\frac{q^2}{Mr_0^3}\left(1+\frac{r_0}{\lambda_D}\right)e^{-r_0/\lambda_D}, \quad (2)$$

where $\lambda_D$ denotes the effective DP Debye radius.[1] The force $F_e=qE(z)$ is due to the electric field $E(z)=-\partial V(z)/\partial z$; the potential $V(z)$ is obtained by solving Poisson's equation, taking into account the sheath potential and also (in a more sophisticated description) the wake potential generated by supersonic ion flows towards the electrode.[17] The potential $V(z)$ thus obtained, actually a *nonlinear* function of $z$, can be developed around the equilibrium position $z_0$ as

$$V(z)\approx V(z_0)+V_{(1)}\delta z+\tfrac{1}{2}V_{(2)}(\delta z)^2+\tfrac{1}{6}V_{(3)}(\delta z)^3+\mathcal{O}[(\delta z)^4], \quad (3)$$

obviously $V_{(j)}\equiv\partial^j V(z)/\partial z^j|_{z=z_0}$; the electric force, therefore, reads

$$F_e(z)\approx F_e(z_0)+\gamma_{(1)}\delta z+\gamma_{(2)}(\delta z)^2+\gamma_{(3)}(\delta z)^3+\mathcal{O}[(\delta z)^4],$$

where all coefficients are defined via the derivatives of $V(z)$, i.e., $\gamma_{(j)}\equiv-q(1/j!)V_{(j+1)}$. The zeroth-order term balances gravity at (and actually defines the value of) $z_0$, viz., $F_e(z_0)-Mg=0$, while $-\gamma_{(1)}=qV_{(2)}=\gamma\equiv M\omega_g^2$ is the effective width of the potential well; the value of the gap frequency $\omega_g=\omega(k=0)$ may either be evaluated from first theoretical principles[8] or determined experimentally,[13] and is typically of the order of $\omega_g/2\pi\approx 20$ Hz in laboratory experiments. Collisions with neutrals and dust charge dynamics are omitted, at a first step, in this simplified model.

Retaining only the linear contribution and considering phonons of the type, $x_n=A_n\exp[i(knr_0-\omega t)]+c.c.$, we obtain an optical-mode-like dispersion relation

$$\omega^2=\omega_g^2-4\omega_0^2\sin^2\left(\frac{kr_0}{2}\right), \quad (4)$$

where $\omega$ and $k=2\pi/\lambda$ denote, respectively, the wave frequency and the wavenumber. We will not go into further details concerning the linear regime, since it is sufficiently covered in Refs. 6–10. Let us now see what happens if the *nonlinear* terms are retained.

For analytical tractability, we shall limit ourselves to a quasi-continuum limit, by considering an amplitude which varies over a scale $L$ which is significantly larger than the inter-grain distance $r_0$ (i.e., $L/r_0\ll 1$). Equation (1) takes the form

$$\frac{d^2u_n}{dt^2}+c_0^2(2\delta u_n-u_{n-1}-u_{n+1})+\omega_g^2u_n+\alpha u_n^2+\beta u_n^3=0, \quad (5)$$

where we set $\delta z\equiv u(x,t)$ for simplicity; $c_0=\omega_0 r_0$ is a characteristic propagation speed related to the interaction [e.g., Debye-type, see (2)] potential; the nonlinearity coefficients $\alpha$, $\beta$ are related to the anharmonicity of the electric potential, viz.,

$$\alpha=-\frac{\gamma_{(2)}}{M}\equiv\frac{qV_{(3)}}{2M}, \quad \beta=-\frac{\gamma_{(3)}}{M}\equiv\frac{qV_{(4)}}{6M}. \quad (6)$$

Remember that inter-grain interactions are *repulsive*, hence the difference in structure from the nonlinear Klein–Gordon equation used to describe one-dimensional oscillator chains. "Phonons" in this chain are stable only in the presence of the electric field (i.e., for $\gamma\neq 0$).

We now proceed by considering small-amplitude oscillations of the form

$$u=\epsilon u_1+\epsilon^2u_2^2+\cdots,$$

at each lattice site. Introducing multiple scales in time and space, i.e., $X_n=\epsilon^n x$, $T_n=\epsilon^n t$ $(n=0,1,2,\ldots)$, we develop the derivatives in Eq. (5) in powers of the smallness parameter $\epsilon$ and then collect terms arising in successive orders. The equation thus obtained in each order can be solved and substituted to the subsequent order, and so forth. This reductive perturbation technique is a standard procedure for the study of the nonlinear wave propagation (e.g., in hydrodynamics, in nonlinear optics, etc.) often used in the description of localized pulse propagation, prediction of instabilities, etc.[22,23] This procedure leads to a solution of the form

$$u(x,t)=\epsilon[Ae^{i(kx-\omega t)}+c.c.]+\epsilon^2\alpha\left[-\frac{2|A|^2}{\omega_g^2}+\frac{A^2}{3\omega_g^2}e^{2i(kx-\omega t)}+c.c.\right]+\mathcal{O}(\epsilon^3), \tag{7}$$

where $c.c.$ denotes the complex conjugate; recall that $\omega$ obeys the dispersion relation (4).

The slowly varying amplitude $A=A(X_1-v_gT_1)$ moves at the (negative) group velocity $v_g=d\omega/dk=-\omega_0^2r_0\sin kr_0/\omega$ in the direction opposite to the phase velocity; this *backward* wave has been observed experimentally: See the discussion in Ref. 14. The amplitude $A$ obeys a Nonlinear Schrödinger Equation (NLSE) of the form

$$i\frac{\partial A}{\partial T}+P\frac{\partial^2 A}{\partial X^2}+Q|A|^2A=0, \tag{8}$$

where the "slow" variables $\{X,T\}$ are $\{X_1-v_gT_1,T_2\}$, respectively. The dispersion coefficient $P$, which is related to the curvature of the phonon dispersion relation (4) as $P=(d^2\omega/dk^2)/2$, reads

$$P=-\frac{c_0^2\omega_0^2}{4\omega^3}\left[2\left(\frac{\omega_g^2}{\omega_0^2}-2\right)\cos(kr_0)+\cos(2kr_0)+3\right], \tag{9}$$

and the nonlinearity coefficient

$$\boxed{Q=\frac{1}{2\omega}\left(\frac{10\alpha^2}{3\omega_g^2}-3\beta\right)}=\frac{1}{2M\omega}\left[-\frac{10\gamma_{(2)}^2}{3\gamma_{(1)}}+3\gamma_{(3)}\right] \tag{10}$$

is related to the electric field nonlinearity considered above. Notice that the sign of $P$ depends on the ratio $\omega_g/\omega_0\equiv\lambda>0$, which appears naturally as an order parameter; see for instance in (4) that $\lambda\geqslant 2$ is a stability criterion (necessary for $\omega$ to be real in the whole range of the first Brillouin zone). For long wavelength values, $P\approx-(c_0^2\omega_g^2)/(2\omega^3)<0$, given the parabolic form of $\omega(k)$ close to $k=0$ (continuum case). In the general (discrete) case, we see that the coefficient $P$ becomes positive at some critical value of $k$, say $k_{\rm cr}$, inside the first Brillouin zone. Some simple algebra shows that the zero-dispersion point $k_{\rm cr}$ satisfies the relation: $\cos(k_{\rm cr}r_0)=(2-\lambda^2+\lambda\sqrt{\lambda^2-4})/2$, for $\lambda>2$ $(0<k_{\rm cr}r_0<\pi)$; otherwise, for $\lambda<2$, $P$ remains negative everywhere.

In a generic manner, a modulated wave whose amplitude obeys the NLS equation (8), is stable (unstable) to perturbations if the product $PQ$ is negative (positive). To see this, one may first check that the NLSE accepts the monochromatic solution (Stokes' wave) $A(X,T)=A_0e^{iQ|A_0|^2T}+c.c.$ The standard (linear) stability analysis then shows that a linear modulation with the frequency $\Omega$ and the wavenumber $\kappa$ obeys the dispersion relation

$$\Omega^2(\kappa)=P^2\kappa^2\left(\kappa^2-2\frac{Q}{P}|A_0|^2\right), \tag{11}$$

which exhibits a purely growing mode for $\kappa\geqslant\kappa_{\rm cr}=(Q/P)^{1/2}|A_0|$. The growth rate attains a maximum value of $\gamma_{\max}=Q|A_0|^2$. This mechanism is known as the Benjamin—Feir instability.[22] For $PQ<0$, the wave is modulationally stable, as evident from (11).

One now needs to deduce the sign of $Q$, given by (10), in order to determine the stability profile of the TDL oscillations. In fact, given the above definitions of the parameters $\omega$, $\alpha$, $\beta$, one easily finds that $Q$ is related to the (derivatives of the) electric potential $V(z)$ via

$$Q=\frac{q}{4M\omega}\left[\frac{5V_{(3)}^2}{3V_{(2)}}-V_{(4)}\right]=\frac{\omega_g^2}{4\omega}\left[\frac{5V_{(3)}^2}{3V_{(2)}^2}-\frac{V_{(4)}}{V_{(2)}}\right]. \tag{12}$$

The exact form of the potential $V(z)$ may be obtained from *ab initio* calculations or by experimental data fitting. For instance, in Ref. 13, the dust grain potential energy $\mathcal{U}(z)=qV(z)$ was reconstructed from experimental data as

$$\mathcal{U}(z)\simeq M\omega_0^2[-0.9\delta z+\tfrac{1}{2}(\delta z)^2-\tfrac{1}{3}0.5(\delta z)^3+\tfrac{1}{4}0.07(\delta z)^4+\cdots], \tag{13}$$

which is Eq. (9) in Ref. 13; upon simple inspection from (3), we obtain $V_{(3)}/V_{(2)}=-1$, $V_{(4)}/V_{(2)}=0.42$, so the value of $Q$ is positive, as may be checked from (12). Therefore, the transverse oscillation considered in Ref. 13 would propagate as a *stable* wave, for large wavelength values $\lambda$. However, for shorter wavelengths, the coefficient $P=\omega''(k)/2$—as defined in (4)—may become positive (and so will the product $PQ$, in this case); the TDL wave may thus be potentially unstable. These results may *a priori* be checked experimentally.

A final comment concerns the possibility of the existence of localized excitations related to transverse dust-lattice waves. It is known that the NLSE (8) supports pulse-shaped localized solutions (envelope solitons) of the bright $(PQ>0)$ or dark/gray $(PQ<0)$ type.[38,39] The former (continuum breathers) are

$$A=(2D/PQ)^{1/2}\,\mathrm{sech}[(2D/PQ)^{1/2}(X-v_eT)]\times\exp[iv_e(X-v_cT)/2P]+c.c., \tag{14}$$

where $v_e$ $(v_c)$ is the envelope (carrier) velocity and $D=(v_e^2-2v_ev_c)/(4P^2)$; they may occur and propagate in the lattice if a sufficiently short wavelength is chosen, so that the product $PQ$ is positive. We note that the pulse width $L$ and the amplitude $\rho$ satisfy $L\rho\sim(|P/Q|)^{1/2}=\text{const}$.

For $PQ<0$, we have the gray envelope soliton[38]

$$A=\rho_1\{1-a^2\,\mathrm{sech}^2\{[X-(v_e+2\alpha)T]/L_1\}\}^{1/2}\exp(i\sigma), \tag{15}$$

where $\sigma=\sigma(X,T)$ is a nonlinear phase correction to be determined. This excitation represents a localized region of negative wave density (a void), with finite amplitude $(1-a)\rho_1$ at $X=0$; $0\leqslant a\leqslant 1$). Again, the pulse width $L_1=(|P/Q|)^{1/2}/(a\rho_1)$ is inversely proportional to the amplitude $\rho_1$. Notice the (dimensionless) parameter $a$, which regulates the depth of the excitation. For $a=1$, one obtains a dark envelope soliton, which describes a localized density hole, characterized by a vanishing amplitude at $\zeta=0$. The latter excitations (of gray/dark type), yet apparently privileged in the continuum limit (where $PQ<0$), are rather physically irrelevant in our (infinite chain) model, since they correspond to an infinite energy stored in the lattice. Never-

theless, their existence locally in a finite-sized chain may be considered (and possibly confirmed) either numerically or experimentally.

In conclusion, we have shown that the modulational instability is, in principle, possible for transverse DP lattice waves. Long wavelength modes seem to ensure wave stability, while shorter wavelength modes may be modulationally unstable. The existence of localized excitations and the occurrence of the modulational instability rely on the same criterion, which needs to be thoroughly examined for a given exact form of the sheath electric potential. These results may be investigated and will hopefully be confirmed by appropriate experiments. Of course, for a more complete description, one has to take into account certain factors that are ignored in this simple model, viz., collisions with neutral particles, dust charge variations and, eventually, transverse to longitudinal mode coupling. The effect of inter-layer coupling may also be investigated. Work in this direction is in progress and the results will be reported soon.

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