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Weakly Nonlinear Effects Associated with Transverse Oscillations in Dusty Plasma Crystals

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Abstract

We study the amplitude modulation of transverse dust lattice waves (TDLW) propagating in a single- and double-layer dusty plasma (DP) crystal. It is shown that a modulational instability mechanism, which is related to an intrinsic nonlinearity of the sheath electric field, may occur under certain conditions. Possibility of the formation of localized excitations (envelope solitons) in the dusty plasma crystal is discussed.

1. Introduction

Wave propagation in nonlinear dispersive media is known to be characterized by amplitude modulation, a manifestation of the nonlinearity coming into play once one departs slightly from the usual small-amplitude (linear) hypothesis. This generic mechanism leads to harmonic generation, due to the carrier wave self-interactions; the wave amplitude may thus undergo modulation (Benjamin–Feir) instability (MI) and eventually collapse [1,2]. Alternatively, this mechanism may favor energy localization via the spontaneous formation of localized excitations (envelope solitons, breathers), as already known in solid state physics (atomic lattices) [3,4] and in a variety of other contexts [2,5]. These effects are effectively modeled by nonlinear theories, based on Nonlinear Schrödinger-type (NLS) equations, so, remarkably, conditions for MI and pulse formation can be exactly formulated in terms of the system's intrinsic physical parameters (affecting the dispersion and nonlinearity laws).

In the context of the plasma wave theory, this nonlinear mechanism has been investigated since long ago [6,7]. In dusty (or complex) plasmas (DP), in particular, new electrostatic wave modes arise [8,9], whose modulation has been studied recently [10–12]; instability conditions were shown to depend strongly on modulation obliqueness, dust concentration [11] and ion temperature [12]. However, to the best of our knowledge, no such study has been carried out, from first principles, in the case of (strongly coupled) DP crystals [13,14]: these strongly-coupled horizontal periodic arrangements of dust grains, typically formed in the sheath region in discharge experiments and suspended above the negative electrode, due to a balance between the (upward) electric forces and gravity, are known to support low-frequency acoustic (in-plane, longitudinal or transverse), as well as optical-mode-like transverse (off-plane) oscillations, whose linear regime has been thoroughly studied both experimentally and theoretically [9].

This brief report is devoted to a study of the nonlinear modulation of transverse (vertical) dust lattice waves (TDLW) propagating in a one-dimensional (1d) dust crystal. The dust grain charge Q and the mass M are assumed to be constant, for simplicity. In principle, nonlinearities in lattice waves may arise as a result of: (*i*) electrostatic inter-grain interactions (Debye-type or else [15]), (*ii*) longitudinal-to-transverse mode couplings, and (*iii*) the form of the sheath electric field. The former two possibilities will be addressed in a forthcoming work. Focusing on the latter, we aim at pointing out:

– the possibility of occurrence of the modulational instability of the TDLW;
– the possibility of the formation of localized structures due to a mutual compensation between the nonlinearity and dispersion, and
– the influence of the layer coupling on the stability profile of the crystal.

2. Linear oscillations—relation to previous results

Transverse motion in a dust crystal (lattice constant r_0) obeys the equation

$$\frac{d^2\delta z_n}{dt^2} = \omega_0^2(2\delta z_n - \delta z_{n-1} - \delta z_{n+1}) + \frac{F_e}{M} - g, \tag{1}$$

where $\delta z_n = z_n - z_0$ denotes the small displacement of the nth grain around the equilibrium position z_0, in the transverse direction ($z-$), propagating in the longitudinal ($x-$) direction. Only the first nearest neighbor interactions between dust grains are retained here. The characteristic frequency ω_0 is related to the inter-grain interaction potential $\phi(r)$, so that

$$\omega_0^2 = -\frac{Q}{Mr_0}\left.\frac{\partial\phi(r)}{\partial r}\right|_{r=r_0}, \tag{2}$$

e.g.,

$$\omega_0^2 = \frac{Q^2}{Mr_0^3}\left(1 + \frac{r_0}{\lambda_D}\right)e^{-r_0/\lambda_D} \tag{3}$$

in the case of the Debye–Hückel interaction potential. Solving Poisson's equation, one obtains the electric field,

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which is due to the sheath potential *and* (in a more complete picture) to the wake potential generated by supersonic ion flow towards the electrode [15]. The total field $E(z)$, actually a *nonlinear* function of z, can be developed around the equilibrium position; the electric force therefore reads

$$F_e(z) \approx F_e(z_0) + \gamma_{(1)}\delta z + \gamma_{(2)}(\delta z)^2 + \gamma_{(3)}(\delta z)^3 + \mathcal{O}((\delta z)^4);$$

where all coefficients are appropriately defined via derivatives of the exact field form [16]. The zeroth-order term $F_e(z_0)$ balances gravity Mg at (and actually defines) the equilibrium position z_0, while $-\gamma_{(1)} = \gamma \equiv M\omega_g^2$ is the effective width of the potential well, which can be determined experimentally [17]. We note that collisions of charged dust particles with neutrals and the dust charge fluctuation dynamics are omitted in our simplified model.

Considering phonons of the type: $x_n = A_n \exp[\mathrm{i}(knr_0 - \omega t)] + c.c.$ and retaining only the linear part in (1), we obtain an optical-mode-like dispersion relation

$$\omega^2 = \omega_g^2 - 4\omega_0^2 \sin^2 \frac{kr_0}{2}. \tag{4}$$

We will not go into further details concerning the linear regime, since it is covered in previous papers. Let us examine what happens if the *nonlinear* terms are retained.

3. Wave modulation—a nonlinear Schrödinger equation

For simplicity, we shall limit ourselves to the continuum limit, considering wavelengths λ significantly larger than the inter-grain distance r_0 (i.e. $kr_0 \ll 1$). With all the above considerations, Eq. (1) takes the form

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + c_0^2\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \omega_g^2 u + \alpha u^2 + \beta u^3 = 0, \tag{5}$$

where we set $\delta z \equiv u(x, t)$ for simplicity; $c_0 = \omega_0 r_0$ is a characteristic propagation speed related to the interaction potential via (2); the nonlinear coefficients α, β are related to the electric field: $\alpha = -\gamma_{(2)}/M$, $\beta = -\gamma_{(3)}/M$. Notice the difference from the nonlinear Klein–Gordon equation used to describe one-dimensional oscillator chains (spring models); inter-particle interactions are *repulsive*, here. Phonons in this chain are stable only in the presence of the electric field (i.e., for $\gamma \neq 0$).

We now consider oscillations of the form

$$u = \varepsilon u_1 + \varepsilon^2 {u_2}^2 + \cdots$$

at each site. Assuming the existence of multiple scales in time and space, i.e., $X_n = \varepsilon^n x$, $T_n = \varepsilon^n t$ ($n = 0, 1, 2, \ldots$), we develop the derivatives in (5) in powers of the smallness parameter ε and then collect the terms arising in successive orders. The equation thus obtained in each order can be solved and substituted into the subsequent order, and so forth. This reductive perturbation technique is a standard procedure in the study of the nonlinear wave propagation in various contexts [1,6].

The procedure outlined above leads to a solution of the type

$$\begin{aligned} u(x, t) &= \varepsilon(A\mathrm{e}^{\mathrm{i}(kx-\omega t)} + c.c.) \\ &\quad + \varepsilon^2\alpha\left(-\frac{2|A|^2}{\omega_g^2} + \frac{A^2}{3\omega_g^2}\,\mathrm{e}^{2\mathrm{i}(kx-\omega t)} + c.c.\right) \\ &\quad + \mathcal{O}(\varepsilon^3) \end{aligned} \tag{6}$$

where ω obeys a dispersion law of the form

$$\omega^2 = \omega_g^2 - c_0^2 k^2, \tag{7}$$

which is (4) linearized around $k \approx 0$.

The slowly-varying amplitude $A = A(X_1 - v_g T_1)$ moves at the (negative) group velocity $v_g = \mathrm{d}\omega/\mathrm{d}k = -c_0^2 k/\omega$, i.e., in the direction opposite to the phase velocity (this so called backward wave has been observed experimentally: see the discussion in Ref. [18]); it is found to obey a *Nonlinear Schrödinger Equation* (NLSE) of the form

$$\mathrm{i}\frac{\mathrm{d}A}{\mathrm{d}T} + P\frac{\mathrm{d}^2A}{\mathrm{d}X^2} + Q|A|^2 A = 0, \tag{8}$$

where the "slow" variables $\{X, T\}$ are $\{X_1 - v_g T_1, T_2\} = \{\varepsilon(x - v_g t), \varepsilon^2 t\}$. The *dispersion coefficient* P is related to the curvature of the phonon dispersion curve (7)

$$P = \frac{1}{2}\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2} = -\frac{c_0^2\omega_g^2}{2\omega^3}, \tag{9}$$

and the *nonlinearity coefficient* Q is related to electric field nonlinearities

$$Q = \frac{1}{2\omega}\left(\frac{10\alpha^2}{3\omega_g^2} - 3\beta\right). \tag{10}$$

Notice that $P < 0$, given the parabolic form of $\omega(k)$ [19].

4. Modulational instability

In a generic manner, a modulated wave whose amplitude obeys the NLS equation (8) is unstable to perturbations if $P \cdot Q > 0$, i.e., from (9), (10) if: $10\gamma_{(2)}^2 - 9\gamma_{(1)}\gamma_{(3)} < 0$. To see this, one may first check that the NLSE accepts the monochromatic solution (Stokes' wave):

$$A(X, T) = A_0 \mathrm{e}^{\mathrm{i}Q|A_0|^2 T} + c.c.$$

The standard (linear) stability analysis then shows that a linear perturbation of the frequency Ω and the wavenumber κ will obey

$$\Omega^2(\kappa) = P^2\kappa^2\left(\kappa^2 - 2\frac{Q}{P}|A_0|^2\right), \tag{11}$$

and is therefore expected to grow if

$$\kappa \geq \kappa_{cr} = (Q/P)^{1/2}\,|A_0|$$

at a rate attaining a maximum value of:

$$\sigma_{\max} = Q|A_0|^2$$

until the wave collapses. Nevertheless, if $P \cdot Q < 0$, this will never occur. This mechanism is known as the *Benjamin–Feir instability* [1].

5. Localized excitations

It is known that the NLSE (8) supports pulse-shaped localized solutions (envelope solitons) of the *bright* ($PQ > 0$) or *dark/grey* ($PQ < 0$) type [5,20]. The former (*envelope pulses*, see Fig. 1) are

$$A = A_0 \mathrm{sech}\left(\frac{X - uT}{L}\right) \exp \mathrm{i} \frac{1}{2P}\left[uX - (\Omega + \tfrac{1}{2}u^2)T\right] + c.c., \tag{12}$$

which represent a localized pulse travelling at a speed u and oscillating at a frequency Ω (at rest). The pulse width L is inversely proportional to the (constant) maximum amplitude A_0, viz. $L = (2P/Q)^{1/2}/A_0$. For values of L comparable to the envelope wavelength (see Fig. 1(b)), this excitation may be seen as a continuum analogue of the (discrete) breathing modes studied in molecular chains [22] and, possibly, also existing in a DP crystal.

The dark/grey-type excitations (*holes*) obtained for $PQ < 0$ [5,20] are physically irrelevant in this (infinite chain) model, since they correspond to an infinite energy stored in the lattice.

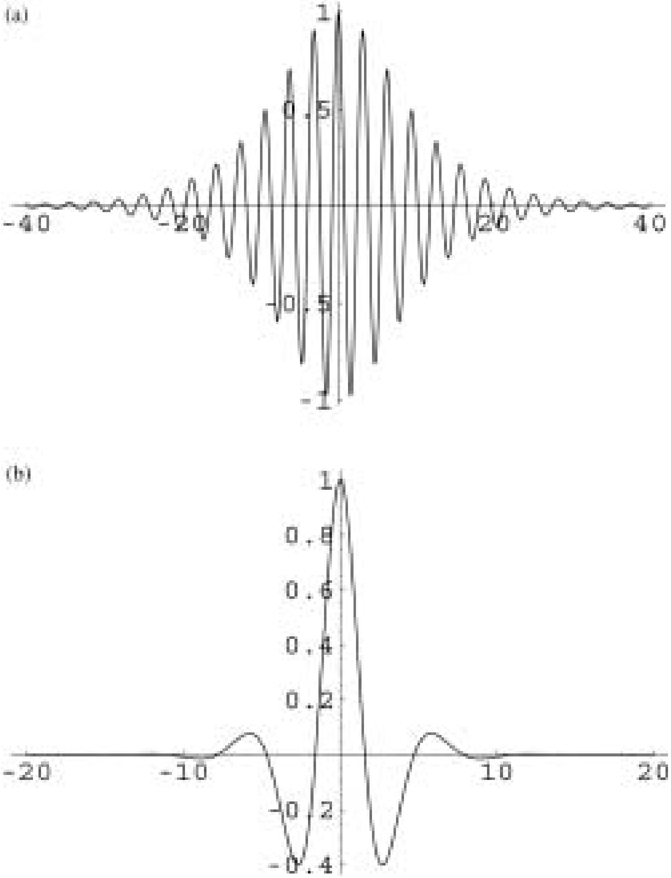

Fig. 1. Bright type (pulse) soliton solutions of the NLS equation (8) (with $PQ > 0$) for two different parameter sets. The second type is reminiscent of the (discrete) breathing modes, encountered in solid state physics [22].

6. Coupled DP lattices

The above picture is strongly modified if a set of coupled DP lattices is considered. For two identical such coupled chains, the equations of motion for the lower (upper) particles 1 (2) read

$$\begin{aligned} M\frac{\mathrm{d}^2\delta z_{1,n}}{\mathrm{d}t^2} &= M\omega_0^2(2\delta z_{1,n} - \delta z_{1,n-1} - \delta z_{1,n+1}) - M\omega_g^2\delta z_{1,n} \\ &\quad + \Gamma_{11}(\delta z_{2,n} - \delta z_{1,n}) + \Gamma_{12}(\delta z_{2,n} - \delta z_{1,n})^2, \\ M\frac{\mathrm{d}^2\delta z_{2,n}}{\mathrm{d}t^2} &= M\omega_0^2(2\delta z_{2,n} - \delta z_{2,n-1} - \delta z_{2,n+1}) - M\omega_g^2\delta z_{2,n} \\ &\quad + \Gamma_{21}(\delta z_{2,n} - \delta z_{1,n}) + \Gamma_{22}(\delta z_{2,n} - \delta z_{1,n})^2, \end{aligned} \tag{13}$$

where Γ_{ij} are nonlinear functions of the inter-chain distance d, related to the electric potential $\Phi_1(z)(\Phi_2(z))$ felt by the lower (upper) grains due to their upper (lower) counterparts; in particular

$$\Gamma_{11} = Q\frac{\mathrm{d}^2\Phi_1(|z|)}{\mathrm{d}|z|^2}\bigg|_{|z|=d}, \qquad \Gamma_{21} = -Q\frac{\mathrm{d}^2\Phi_2(|z|)}{\mathrm{d}|z|^2}\bigg|_{|z|=d}.$$

Note that the two potentials are *not* symmetric: $\Phi_2(z)$ acting on the upper particles due to the lower ones is a simple Debye–Hückel-type potential, but $\Phi_1(z)$ felt by the lower particles due to the upper ones is modified by downwards ion flow. The dispersion relation obtained in the linear limit now consists of two distinct dispersion branches, both given by (3) with gap frequencies equal to $\omega_{g,1} = \omega_g$ and $\omega_{g,2} = \sqrt{\omega_g^2 + \Gamma_{11}/M + \Gamma_{21}/M}$.

The perturbative analysis described above is now more tedious but straightforward. Omitting lengthy expressions (to be reported elsewhere [21]), we shall outline the main qualitative aspects of the study.

The group velocity $v_g = \omega'(k)$ in both lattices is the same as in the single-layer case. The first harmonics in $\sim\varepsilon^1$, as well as second- and zeroth-order corrections arising in $\sim\varepsilon^2$ are now coupled, due to layer couplings. Suppression of secular terms in order ε^3 now results in two coupled NLS (CNLS) equations of the form

$$\begin{aligned} \mathrm{i}\frac{\mathrm{d}A}{\mathrm{d}T} + P\frac{\mathrm{d}^2A}{\mathrm{d}X^2} + Q_{11}|A|^2A + Q_{12}|B|^2A = 0, \\ \mathrm{i}\frac{\mathrm{d}B}{\mathrm{d}T} + P\frac{\mathrm{d}^2B}{\mathrm{d}X^2} + Q_{21}|A|^2B + Q_{22}|B|^2B = 0, \end{aligned} \tag{14}$$

where A, B denote the first harmonic amplitudes in the two lattices; the (slow) variables are as defined previously. The dispersion coefficient $P = \omega''(k)/2$ is given by (9); the lengthy expressions for the (non-symmetric) nonlinearity matrix elements Q_{ij}, in fact complicated analytic functions related to the interaction potentials Φ_1, Φ_2 [21], are omitted here.

The linear stability analysis around the coupled monochromatic-wave solutions

$$\begin{aligned} A(X,T) &= A_0 e^{\mathrm{i}(Q_{11}|A_0|^2 + Q_{12}|B_0|^2)T} + c.c., \\ B(X,T) &= B_0 e^{\mathrm{i}(Q_{22}|B_0|^2 + Q_{21}|A_0|^2)T} + c.c. \end{aligned}$$

now results in the dispersion relation

$$\Omega^2 = \tfrac{1}{2}\Big[\big(\Omega_{12}^2 + \Omega_{22}^2\big) + \sqrt{\big(\Omega_{11}^2 - \Omega_{22}^2\big) + 4\Omega_{12}^2\Omega_{21}^2}\,\Big], \qquad (15)$$

where

$$\Omega_{11}^2(\kappa) = P^2\kappa^2\left(\kappa^2 - 2\frac{Q_{11}}{P}|A_0|^2\right),$$

$$\Omega_{22}^2(\kappa) = P^2\kappa^2\left(\kappa^2 - 2\frac{Q_{22}}{P}|B_0|^2\right),$$

$$\Omega_{ij}^2(\kappa) = -2PQ_{ij}|A_0||B_0|\kappa^2, \qquad (i \neq j = 1, 2) \qquad (16)$$

(κ is the perturbation wavenumber); we see that, for vanishing cross-coupling Q_{ij} terms, the single wave perturbation dispersion relation (11) is recovered.

The investigation of the conditions for Ω to possess an imaginary part are now more perplex [21]. An enlarged instability region in κ values is obtained, in terms of Q_{ij}. One highlight of the analysis is that *a stable (single-layer) wave mode* (i.e., for $PQ_{ii} < 0$) *may become unstable due to layer couplings*.

Of course, the next step in this investigation should consist of assuming an explicit form for the electrostatic potentials $\Phi_1(z), \Phi_2(z)$ and deriving exact expressions for the coefficients in the CNLS equations (14) above. The conditions for instability will then be explicitly formulated in terms of intrinsic plasma parameters.

7. Conclusions

We have seen seen that:

(*i*) *modulational instability* is, in principle, possible in transverse DP lattice waves;
(*ii*) instability is potentially *enhanced* by inter-layer coupling;
(*iii*) the existence of localized excitations *and* the occurrence of the modulational instability rely on the same criterion, which needs to be thoroughly examined in terms of the dust particle interaction laws; i.e., for a given exact form of interaction potential(s). The description will then be less abstract and may hopefully be experimentally investigated.

Of course, for a more complete description, one has to take into account a number of factors ignored in this simple model: collisions between dust particles and neutrals, dust charge variations, inter-grain interaction nonlinearity and, very important, transverse to longitudinal mode couplings. Work in this direction is in progress and the results will be reported soon.

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