Modulated Wave Packets and Envelope Solitary Structures in Complex Plasmas

Ioannis Kourakis and Padma K. Shukla

Abstract—A theoretical study is presented of the nonlinear amplitude modulation of waves propagating in unmagnetized plasmas contaminated by charged dust particles. Distinct well-known dusty plasma modes are explicitly considered, namely, the dust-acoustic wave, the dust-ion acoustic wave, and transverse dust-lattice waves. Using a multiple-scale technique, a nonlinear Schrödinger-type equation is derived, describing the evolution of the wave amplitude. A stability analysis reveals the possibility for modulational instability to occur, possibly leading to the formation of different types of envelope-localized excitations (solitary waves), under conditions which depend on the wave dispersion laws and intrinsic dusty plasma parameters.

Index Terms—Complex plasma, electrostatic plasma waves, envelope solitons, modulational instability, nonlinear Schrödinger (NLS) equation.

I. INTRODUCTION

LARGE ensembles of charged particles (plasmas) contaminated with massive heavily charged dust particles, known as complex or dusty plasmas (DP), have recently attracted a constantly increasing interest among researchers, due to their wide occurrence in space and laboratory plasmas and the wealth of novel physics involved in their description [1], [2]. The characteristics of known harmonic modes [3], [4] propagating in a plasma are strongly modified by the presence of dust [1], [2]; for instance, the frequency range of ion-acoustic waves (IAW) [3], [4] is strongly modified in the presence of dust, giving rise to dust-ion acoustic waves (DIAW) [5]. Furthermore, brand new low-frequency modes have been shown to exist in dusty plasma, either as electrostatic oscillations related to dust grain inertia, in weakly coupled (“gaseous”) complex plasmas, e.g., the dust-acoustic wave (DAW) [1], [6]–[8], or as periodic excitations propagating in strongly coupled (“crystalline phase”) DP arrangements (“quasi-lattices”), e.g., dust-lattice waves (DLW) [1], [9]–[14]. Interestingly, the phase velocity of these dusty plasma modes lies far from both electron and ion thermal velocities \( v_{\text{th,e}/i} = (T_{e/i}/m_{e/i})^{1/2} \), thus ruling out the occurrence of Landau damping, known to prevail over electrostatic wave propagation in electron–ion (e–i) plasma [1]–[4].

It is now commonly known that linear oscillations, yet indispensable as a first approach to most dynamical problems, are rarely left to lead a quiet undisturbed life. Once the slightest important departure from equilibrium is considered, omnipresent nonlinearity mechanisms come into play, initially manifesting themselves via harmonic generation due to carrier wave self-modulation, at a first step, then possibly inducing instability to external perturbations, and eventually leading to wave mode breakup (collapse); in many situations, the interplay between nonlinearity and wave dispersion may lead to the formation of nonequilibrium stationary localized structures. This generic picture is known to be valid in physical contexts as diverse as hydrodynamics, condensed matter, nonlinear optics, signal transmission lines, etc. [16], [17].

Plasma physics has, rather not surprisingly, always provided an excellent test-bed for nonlinear modulation theories [18]–[31] and dusty plasma has been no exception [32]–[39], [56]. Reviewing a few noteworthy results, electron plasma modes have been shown to be stable against parallel modulation [20]; the ion plasma modes do as well, yet only for perturbations below a specific wavenumber threshold [24]. Ion acoustic modes are stable to parallel modulation [18]–[22] for wavelengths above a specific threshold, which is seen to increase if one takes into account finite-temperature effects [23]–[25] or oblique amplitude modulation [26]–[28]. These results have been confirmed by kinetic-theoretical studies [29]–[31], for ion–acoustic waves in e–i plasmas. In dusty plasma, the amplitude modulation of the DP acoustic modes has been investigated in [32]–[36]; similar studies have been carried out for oscillations in DP crystals [37]–[39], [56].

This paper aims at investigating the role of nonlinear amplitude modulation mechanism in dusty plasma modes, first, and also questioning, in a generic manner, the existence of envelope excitations (solitons) related to these modes. By making use of a multiple space and time scale technique [18], [19], a nonlinear Schrödinger-type equation (NLSE) is derived, describing the evolution of the wave envelope. The conditions for modulational instability are discussed, pointing out their dependence on the carrier wavenumber \( k \), the modulation angle \( \theta \), and dust concentration \( n_d \). Finally, the existence of different types of envelope dusty plasma solitons is anticipated. In order to keep a low level of complexity in this qualitative description, known DP dissipative mechanisms like collisions with neutrals, ion drag, and dust charge variations are omitted; these effects are left for consideration in forthcoming work.
II. DUST–ACOUSTIC WAVES

The DAW, theoretically predicted in [6] and later experimentally observed [7], [8], is a very low-frequency purely DP mode (e.g., absent without dust) representing inertial dust grain oscillations against a thermalized background of electrons and ions, which provide the necessary restoring force. The DAW phase velocity is much smaller than both electron and ion thermal speeds and its frequency is below the dust plasma frequency \( \omega_{pD} \).

A. Model Equations

In order to formulate a simple model for DAW propagation, we consider a three component collisionless unmagnetized dusty plasma consisting of electrons (mass \( m_e \), charge \( e \)), ions (mass \( m_i \), charge \( q_i = \pm Ze \)), and heavy dust particulates (mass \( m_d \), charge \( q_d = \pm Z_d e \)), henceforth denoted by \( e, i, d \) respectively. Dust mass and charge will be taken to be constant, for simplicity.

The dust component moment-Poisson system of (reduced) evolution equations is

\[
\begin{align*}
\frac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{u}) &= 0, \\
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} &= -s \nabla \phi - \frac{\sigma}{n} \nabla p, \\
\frac{\partial p}{\partial t} + \mathbf{u} \cdot \nabla p &= -\gamma p \nabla \phi + \nabla q \mathbf{u}
\end{align*}
\]

(1)

where all quantities are dimensionless; \( n, \mathbf{u} \) and \( p \), respectively, denote the (normalized) dust density \( n_d/n_d^{(0)} \) and mean velocity \( \mathbf{u}_d/\mathbf{u}_d^{(0)} = [(m_d/(k_B T_e))]^{1/2} \mathbf{u}_d \) and pressure \( p_d/p_0 = p_d/(n_d^{(0)} k_B T_e) \); \( \gamma = (f + 2)/f \) is the ratio of specific heats (\( f \) is the number of degrees of freedom). For example, \( \gamma = 3 \) in the adiabatic one-dimensional (1-D) case; space and time are scaled over the DP effective Debye length \( \lambda_{D,\alpha}^{eff} = (\lambda_{D,e}^2 + \lambda_{D,i}^2)^{-1/2} \) (where \( \lambda_{D,\alpha} = (k_B T_e/4\pi n_{\alpha}^{(0)} e^2)^{1/2}, \alpha = e, i \)) and the inverse DP plasma frequency \( \omega_{pD}^{n} = (4\pi n_{\alpha}^{(0)} e^2/m_\alpha)^{1/2} \), respectively. The (reduced) electric potential \( \phi = Z_d e \Phi / (k_B T_e) \) obeys Poisson’s equation: \( \nabla^2 \Phi = -4\pi \sum \mathbf{k}_q n_q, \) which here it takes the form

\[
\nabla^2 \phi = -\alpha \phi^2 + \alpha' \phi^3 - \beta \phi (n - 1)
\]

(2)

by linearizing around the Maxwillian state assumed for both electrons and ions, e.g., \( n_e \approx n_{e,0} e^{e V/k_B T_e}, n_i \approx n_{i,0} e^{-e V/k_B T_i} ; T_e/T_i \) is the temperature of species \( \alpha = e, i \) (\( k_B \) is the Boltzmann constant). The dimensionless parameters are: \( \alpha = (1/2Z^2)(Z^2_{e}(T_e/T_i)^{1/2} n_{e,0}/n_{i,0} - 1)/(Z^2_{i}(T_i/T_e)^{1/2} n_{i,0}/n_{e,0} + 1), \) \( \alpha' = (1/6Z^2)(Z^2_{e}(T_e/T_i)^{1/2} n_{e,0}/n_{i,0} + 1)/(Z^2_{i}(T_i/T_e)^{1/2} n_{i,0}/n_{e,0} + 1), \) and \( \beta = \lambda_{D,\alpha}^{eff} \mathbf{u}_d/|\mathbf{u}_d| \). The DA speed is \( |\mathbf{u}_d| \). Alternatively, one may define these parameters in terms of typical dust parameters, e.g., either the ratio \( \mu = n_{e,0} / (Z_d n_{i,0}) \) or \( \delta = (Z_d n_{i,0}) / (Z_d n_{e,0}) \). For \( \mu \ll 1, T_e/T_i \), one has \( \alpha \approx (Z_e/2Z^2_i)(T_e/T_i), \alpha' \approx (Z^2_e/6Z^2_i)(T_e/T_i)^2 = 2/3 \alpha^2 \) and \( \beta \approx (Z^2_e/Z^2_i)(n_{e,0}/n_{i,0})(T_i/T_e) \). Since overall neutrality is assumed at equilibrium \( n_{e,0} - Z_d n_{i,0} - Z_d n_{d,0} = 0 \), one has \( \mu = 1 + s \delta \), so that \( 0 \leq \mu < 1 (\mu > 1) \) corresponds to negative (positive) dust charge,; obviously, \( \mu = 0 \) in the absence of dust.

Notice that the temperature ratio \( \sigma = p_d/(n_d k_B T_e) = T_d/T_e \) will “tag” the effect of the pressure evolution being taken into account in this “warm” dust model [33]. In contrast, e.g., to [36], we see that the “cold” model expressions (9)–(11) therein are readily recovered upon setting \( \sigma = 0 (s = -1, \alpha' = 0) \).

B. Multiple Scales Perturbation Method

Let \( \mathbf{S} \) be the state (column) vector \( (n, \mathbf{u}, \mathbf{p}, \phi)^T \), describing the system’s state at a given position \( \mathbf{r} \) and instant \( t \). We shall consider small deviations from the equilibrium state \( \mathbf{S}^{(0)} = (1, 0, 1, 0)^T \) by taking \( \mathbf{S} = \mathbf{S}^{(0)} + \epsilon \mathbf{S}^{(1)} + \epsilon^2 \mathbf{S}^{(2)} + \cdots = \mathbf{S}^{(0)} + \sum_{n=0}^{\infty} e^n \mathbf{S}^{(n)} \), where \( e \ll 1 \) is a smallness parameter. Following the standard multiple scale (reductive perturbation) technique [18], [19], we shall consider the stretched (slow) space and time variables \( \zeta = \epsilon \mathbf{x} - \lambda \mathbf{t}, \gamma = t^2 (\lambda \in \Re) \). The perturbed states are assumed to depend on the fast scales via the carrier phase \( \theta_1 = \mathbf{k} \cdot \mathbf{r} - \omega t \), while the slow scales enter the argument of the \( j \)-th element’s \( j \)-th harmonic amplitude \( S_{j_d}^{(n)} \), which are to vary along \( x, \) viz., \( S_{j_d}^{(n)} = \sum_{z=-\infty}^{\infty} S_{j_d}^{(n)}(\zeta, \tau) e^{jk_\mathbf{r} - \omega t} \) (where \( S_{j_d}^{(n)} = S_{j_d}^{(n)\ast} \)). The amplitude modulation (along the \( x \) axis) is thus allowed to take place in an oblique direction, with respect to the (arbitrary) propagation direction; accordingly, the wavenumber \( \mathbf{k} \) is \( (k_x, k_y) = (k \cos \theta, k \sin \theta) \). Treating the derivative operators as

\[
\begin{align*}
\frac{\partial}{\partial \zeta} &\rightarrow \frac{\partial}{\partial \zeta} - \epsilon \lambda \frac{\partial}{\partial \lambda} + \epsilon^2 \frac{\partial}{\partial \tau} + \frac{\partial}{\partial \mathbf{k}}, \\
\nabla \cdot &\rightarrow \nabla + \epsilon k \frac{\partial}{\partial \mathbf{k}}, \\
\nabla^2 &\rightarrow \nabla^2 + 2 \epsilon k \frac{\partial^2}{\partial \mathbf{k}^2} + \epsilon^2 \frac{\partial^2}{\partial \mathbf{k}^2} + \epsilon^2 \frac{\partial^2}{\partial \mathbf{k}^2}
\end{align*}
\]

and substituting into the system of evolution equations, one obtains an infinite series in both (perturbation order) \( \epsilon^n \) and (phase harmonic) \( \lambda \). The standard perturbation procedure now consists of solving in successive orders \( e^n \) and substituting in subsequent orders. The calculation, particularly lengthy yet perfectly straightforward, is reported elsewhere [33], so only the essential steppingstones are provided here. The \( (n = 2, l = 1) \) equations determine the first harmonics of the perturbation

\[
\begin{align*}
n^{(1)}_1 &= s + \frac{1}{\beta} \phi^{(1)}_1 = \frac{1}{\gamma} \phi^{(1)}_1 = \frac{1}{\omega} \mathbf{k} \cdot \mathbf{u}_1^{(1)} \\
\gamma_1^{(1)} &= \frac{k}{\omega \sin \theta} u_1^{(1)}, \quad \phi^{(1)}_1 = \frac{k}{\omega \cos \theta} u_1^{(1)} \\
\omega^2 &= \frac{c_d^2}{1 + k^2 \lambda_{D,\alpha}^{eff}} + \gamma_1^{(1)} u_1^{(1)} k^2 
\end{align*}
\]

(3)

and provide the compatibility condition \( \omega^2 = (\beta k^2/k^2 + 1) + \gamma_1^{(1)} k^2 \), which exactly recovers, by restoring dimensions, the standard DAW dispersion relation [1], [6].

\[
\omega^2 = \frac{c_d^2}{1 + k^2 \lambda_{D,\alpha}^{eff}} + \gamma_1^{(1)} u_1^{(1)} k^2.
\]

(4)

Proceeding in the same manner, we obtain the second-order quantities, namely the amplitudes of the second harmonics \( S_{j_d}^{(2)} \) and constant (“direct current”) terms \( S_{j_d}^{(0)} \), as well as a finite contribution \( S_{j_d}^{(1)} \) to the first harmonics; the lengthy expressions,
omitted here for brevity, can be found in [33]. The \((n = 2, l = 1)\) equations provide the compatibility condition:

\[
\lambda = \frac{k}{\omega} \left[ \frac{1}{(1 + k^2)^2} + \gamma \sigma \right] \cos \theta,
\]

\(\lambda\) is, therefore, the group velocity \(v_g(k) = \partial \omega / \partial k_x = \omega(k) \cos \theta\) in the modulation \((x-)\) direction.

\section*{C. Derivation of the Nonlinear Schrödinger Equation}

Proceeding to order \(\sim e^3\), the equations for \(l = 1\) yield an explicit compatibility condition in the form of the Nonlinear Schrödinger equation:

\[
i \frac{\partial \psi}{\partial t} + P \frac{\partial^2 \psi}{\partial x^2} + Q |\psi|^2 \psi = 0
\]

(5)

where \(\psi\) denotes the potential correction \(\phi^{(1)}\). The "slow" variables \(\{z, t\}\) were defined above.

The dispersion coefficient \(P\) is related to the curvature of the dispersion curve as

\[
P = \frac{1/2}{2} \left[ \frac{\partial \omega}{\partial k_x} \right] \left\{ \frac{\partial^2 \omega}{\partial k_x^2} \right\}
\]

(6)

where we have defined

\[
\nu_2 = \frac{3\beta^2 + \gamma \sigma(3 - k^2)(1 + k^2)}{3(3 + \gamma \sigma(1 + k^2)^2)}
\]

(7)

and

\[
\nu_1 = \frac{\beta \beta^2 + \gamma \sigma(1 + k^2)^2}{3 + \gamma \sigma(1 + k^2)^2}
\]

(8)

(see that \(\nu_1, \nu_2 \to 1\) when \(\sigma \to 0\)). Notice the influence of \(\sigma\) on the sign of \(P\).

The nonlinearity coefficient \(Q\) is due to carrier wave self-interaction. Distinguishing different contributions, \(Q\) can be split into five distinct parts, viz., \(Q = \sum_{j=0}^{4} Q_j\), where \(Q_0/2\) is due to the zeroth/second order harmonics and \(Q_1\) is related to the cubic term in (2) with \(\psi = \psi_0 + e^{ikt} + c.c.\) (complex conjugate) by setting \(\psi = \psi_0 + e^{ikt} + c.c.\). Substituting into (5), one thus readily obtains \(\omega^2 = P^2 k^2 \left( k^2 - 2Q / |\psi_{1,0}|^2 \right)\). The wave will thus be stable \((\nu_2 > 0)\) if the product \(PQ\) is negative. However, for positive \(PQ > 0\), instability sets in for wavenumbers below a critical value \(k_{cr} = \sqrt{2Q / |\psi_{1,0}|^2}\), e.g., for wavelengths above a threshold \(\lambda_{cr} = 2\pi / k_{cr}\). Defining the instability growth rate \(\sigma = \text{Im}(\omega) / k\), we see that it reaches its maximum value for \(k = k_{cr} / \sqrt{2}\), viz.

\[
\sigma_{\text{max}} = \text{Im}(\omega) = |\psi_{1,0}|^2 = |Q| |\psi_{1,0}|^2.
\]

(9)

Briefly, we see that the instability condition depends only on the sign of the product \(PQ\), which may be studied numerically, relying on the expressions derived above; see Figs. 1 and 2 where we have chosen a set of representative values \(\alpha = 5 \cdot 10^{-3}, \alpha' = 2 \cdot 3 / 3 \approx 1.6 \cdot 10^{-5}\) and \(\beta = 100\), corresponding to \(Z_d / Z_0 = 10^3\) and \(T_e / T_i = 10\) \((\gamma = 2, \sigma = 1)\).

As predicted above, the wave is always stable for very long wavelengths \((k \ll 1)\). For parallel modulation \((\theta = 0)\), modulational instability sets in for carrier wavenumbers \(k\) beyond a critical value, say \(k_{cr}\). For \(\sigma \neq 0\), a second threshold appears, say \(k'_{cr}\), beyond which the wave becomes stable again. Very short wavelengths are also stable in the "hot" dust model. As anticipated, obliqueness in modulation drastically modifies the stability profile. The instability threshold \(k_{cr}\) is seen to decrease as \(\theta\) varies from zero to some critical value \(\theta_{cr}\); nevertheless, beyond that value \((\text{up to } / \pi / 2)\) the wave remains stable. This is even true for the wavenumber regions where the wave would be unstable to a parallel modulation. The inverse effect is also present. Even though certain \(k\) values correspond to stability for \(\theta = 0\), the same modes may become unstable when
subject to an oblique modulation ($\theta \neq 0$); this is mostly true for long wavelengths (small $k$). These qualitative results, obtained for negative dust, still hold in the case of positive dust ($s = +1$; see Figs. 1(b) and 2(b)), yet the instability region is somewhat narrower. Positive dust appears to slightly favor stability.

E. Nonlinear Excitations

The NLSE (5) is known to possess distinct types of localized constant profile (solitary wave) solutions. It should be stressed that these envelope structures should be distinguished in nature from soliton-type localized excitations, e.g., exact solutions of generic nonlinear equations like, e.g., the Korteweg–de Vries (KdV) equation; see the extensive discussion in [41]. Several types of such localized modes are reviewed in [41]–[43], so we need only briefly outline the method employed to derive the analytical form of those we are interested in and discuss their relevance to our problem.

Following [42] and [43], one may seek a solution of (5) in the form $\psi(\zeta, \tau) = e^{-i k_0 x}e^{i \Omega_0 t}$. Different types of solutions are thus obtained, depending on the sign of the product $PQ$.

For $PQ > 0$, we find the (bright) envelope soliton

$$ \rho = \rho_0 \text{sech}^2 \left( \frac{\zeta - ut}{L} \right), \quad \Theta = \frac{1}{2P} \left[ \rho \zeta - \left( \frac{\Omega + \frac{1}{2} u^2}{2P} \right) \tau \right] $$

For $PQ < 0$, we have the dark envelope soliton

$$ \rho = \rho_1 \left[ 1 - \text{sech}^2 \left( \frac{\zeta - ut}{L'} \right) \right] = \rho_1 \tanh^2 \left( \frac{\zeta - ut}{L'} \right), $$

$$ \Theta = \frac{1}{2P} \left[ \rho \zeta - \left( \frac{1}{2} u^2 - 2PQ \rho \right) \tau \right] $$

representing a localized pulse travelling at a speed $u$ and oscillating at a frequency $\Omega$ (for $u = 0$); see Fig. 4. The pulsewidth $L$ depends on the (constant) maximum amplitude square $\rho_0$ as $L = \sqrt{2P/Q\rho_0}$.

For $PQ < 0$, we have the (gray) envelope soliton

$$ \rho = \rho_2 \left[ 1 - \alpha^2 \text{sech}^2 \left( \frac{\zeta - ut}{L''} \right) \right], $$

$$ \Theta = \frac{1}{2P} \left[ \rho \zeta - \left( \frac{1}{2} V_0^2 - 2P \rho_2 \right) \tau + \Theta_10 \right] $$

$$ - S \sin^{-1} \frac{\alpha \tanh \left( \frac{\zeta - ut}{L''} \right)}{\left[ 1 - \alpha^2 \text{sech}^2 \left( \frac{\zeta - ut}{L''} \right) \right]^{1/2}} $$

which represents a localized negative wave density region.\footnote{These expressions are readily obtained from the ones in by transforming the variables therein into our notation as follows: $x \rightarrow \zeta, s \rightarrow t, \beta_m \rightarrow \rho_0, \alpha \rightarrow 2P, \theta \rightarrow \rho_0, \Omega \rightarrow L, E \rightarrow \Omega, V_0 \rightarrow u$.}

Here, $\Theta_10$ is a constant phase; $S$ denotes the product $S = \text{sign} P \times \text{sign}(u - V_0)$. Notice that not only the (now finite) maximum amplitude $\rho$ but also the pulsewidth $L'' = \sqrt{2P/Q\rho_2}$ now depends on $\alpha$, given by $\alpha^2 = 1 + (1/2PQ)(1/\rho_2(\zeta^2 - V_0^2) \leq 1$, an independent parameter representing the modulation depth ($0 < \alpha \leq 1$); see Fig. 5(b) (where $a = 0.6$, $V_0$ is a real constant which satisfies $[42]$ and [43]. $V_0 - \sqrt{2P/Q\rho_2} \leq u \leq V_0 + \sqrt{2P/Q\rho_2}$; for $V_0 = u$, we have $a = 1$ and thus recover the dark soliton (11).

In conclusion, the wave instability (stability) regions depicted in white (black) in Figs. 1 and 2, in fact, also delimit the $(k, \theta)$ parameter pair values where bright (dark/gray) solutions, e.g., density pulses (holes) may exist. Furthermore, soliton characteristics will depend on the carrier wave dispersion via $P$ and $Q$, for instance, regions with lower values of $P$ (or higher values of $Q$) will support narrower excitations.
III. DUST–ION ACOUSTIC MODE

The DAW [1], [5] is the DP analogue of the well-known ion–acoustic electrostatic wave (IAW) [3], [4], where inertial ions oscillate against a background of thermal electrons and massive dust grains. The DIAW phase velocity \( v_{ph} \) lies between (and far from) the electron and ion thermal (sound) velocities, e.g., \( v_{th} \) \( \ll v_{ph} \ll v_{th} \). Finally, the DIAW frequency is much higher (lower) than the dust (ion) plasma frequency \( \omega_{pd} (\omega_{pe}) \), typically tens of kilohertz in laboratory plasmas; therefore, on the time scale of interest, stationary dust does not participate in the wave dynamics and may be assumed immobile.

A. Model

The moment evolution equations for ions can be cast in a reduced (nondimensional) form identical to the system of (1) (for \( s = 1 \)), by making use of an appropriate scaling [34], [35]; space and time are scaled over the electron Debye length \( \lambda_{De,e} = (k_B T_e/4\pi n_e e^2)^{1/2} \) and the characteristic time-scale \( \lambda_{De,e}/c_s = \omega_{pe}^2 (m_i/m_e) \), respectively, where \( \omega_{pe} \) obviously denotes the electron plasma frequency. The ion density, velocity, and pressure are normalized by: the ion equilibrium density \( n_{i0,e} \), the “sound speed” \( c_s \equiv (k_B T_i/m_i)^{1/2} \), and \( p_i = n_i k_B T_i \), respectively. \( \sigma \) now denotes the temperature ratio \( T_i/T_e \). The system is closed with Poisson’s equation, whose right-hand side is Taylor-developed near equilibrium, e.g., \( \Phi \approx 0 \). Now, combining with the equilibrium charge neutrality requirement and normalizing the electric potential \( \Phi \) over \( k_B T_e/(Z_i e) \), one exactly recovers (2), upon setting \( s = 1, \alpha \to -\tilde{\alpha} \) therein. The characteristic (dimensionless) parameters are \( \tilde{\alpha} = 1/(2Z_i), c_{\alpha} = 1/(6Z_i^2) \) and \( \beta = Z_i^2 n_{i0,e}/n_{e0,e} = Z_i/\mu \).

We see that the dust component manifests its presence through the parameter \( \mu \) defined in the previous section; recall that \( \mu < 1 \) (\( \mu > 1 \)) refers to negative (positive) dust charge. As expected (and indeed verified), previous results for the IAW (in the absence of dust) [21], [25], [26] are recovered for \( \mu = 1 \).

B. Amplitude Modulation—Numerical Analysis

The analytical method adopted for the study of amplitude modulation and harmonic generation in the DIAW case follows the lines described in the previous section. The detailed calculation [34], [35] need not be reproduced here, since the analytical result is, in fact, exactly equivalent to the expressions obtained from the formulas presented in Section II upon setting \( s = 1, \alpha \to -\tilde{\alpha} \). In particular, one thus obtains the NLS (5), which now describes the evolution of the DIAW amplitude, along with new definitions of the dispersion and nonlinearity coefficients, \( P \) and \( Q \), obtained from (6) and (7)–(9), respectively.

As can be checked from the approximate expressions obtained for \( P \) and \( Q \) in the limit \( k \ll k_D \), the wave is always stable for very long wavelengths (\( \lambda \gg \lambda_D \)).\(^3\) For parallel modulation (\( \theta = 0 \)), modulational instability sets in for carrier wavenumbers \( k \) beyond a critical value, say \( k_{cr} \); this was qualitatively expected, given the analogous studies of IAW stability (without dust) [18]–[22] (one is relieved, after such a tedious calculation, to recover exactly the previous result \( k_{cr} = 1.47 k_D \), for \( \mu = 1 \); see Fig. 3(a)). Considering a finite ion temperature \( T_i \) (i.e., \( \sigma \neq 0 \)), a second threshold appears, say \( k_{cr}' \), beyond which the wave becomes stable again (corresponding plots, omitted here for brevity, are qualitatively analogous to Fig. 2). Very short wavelengths are therefore also stable in the “hot” ion model (not true for \( T_i = 0 \), where \( k_{cr}' \to \infty \)) and the instability region shrinks as \( k_{cr}' \) decreases with increasing ion temperature \( T_i \), in agreement with IAW results [23]. The presence of negative dust (\( \mu < 1 \)) results in a narrower stability region at low \( k \) (e.g., lower \( k_{cr}' \)); negative dust therefore destabilizes long wavelength DIAW. A minimum as low as \( k_{cr,min} \approx 0.0285 k_D \) is reached around \( \mu \approx 0.33 \), e.g., for a negative dust charge density as high as \( \delta \approx 0.67 \) times the ion density. On the other hand, positive dust rather favors stability, as it slightly increases the instability threshold \( k_{cr} \) (see [34] and [35] for details).

\(^3\) Substitute into the approximate expressions for \( P, Q \) in the low \( k \) limit (Section II) with the \( [\alpha, \beta, s, \theta] \) parameter assignments in Section III; cf. [21, eq. (41)]; note that the factor 1/3 therein is thus exactly recovered herein for \( \mu = 1, Z_i = 1 \), e.g., setting \( \alpha = -\tilde{\alpha} = -1/2, \alpha' = 1/6, \beta = 1, \sigma = 0 \) (and \( \theta = 0 \)).
As expected from earlier predictions on IAW propagation in e–i plasma [26], obliqueness in modulation modifies the stability profile quite dramatically. The instability threshold \( k_{\text{cr}} \) is seen to decrease as \( \theta \) varies from zero to some critical value \( \theta_{\text{cr}} \) and then goes to infinity for higher angle values. Wide-angle modulation results in absolute stability: cf. Fig. 3(a) [however, this is not true for ultra high negative dust presence, e.g., below \( \mu \approx 0.1 \); see Fig. 3(b)].

Briefly concluding, oblique modulation may stabilize otherwise unstable waves and vice versa. This may also result in a shift in the type of solitary excitations sustained in the plasma, e.g., bright solitons may be destabilized when subject to oblique modulation, giving their place to their dark counterparts as preferred modes or vice versa. See [34] and [35] for a detailed study.

IV. DUST-LATTICE MODES

It is now established, both theoretically and experimentally, that the increased intergrain electrostatic interaction due to the high charge acquired by dust may result in the formation of quasi-crystalline, strongly coupled dust particle periodic arrangements [1]. These dust-lattices (DL), which are typically observed at some levitated horizontal position in the sheath region above the negative electrode in discharge experiments, have been shown [9]–[13] (and experimentally observed in [14] and [15]) to support linear oscillations in both longitudinal [9]–[12] and transverse directions [13]. Because of the intrinsic chain nonlinearity, either related to the electrostatic (Debye-like, typically) interactions, in the horizontal direction, or to the sheath potential form, in the vertical one, one would a priori expect these excitations to be subject to a variety of nonlinear mechanisms which may underlie phenomena observed in DP crystals, e.g., instabilities, phase transitions (“melting”), etc. Yet, quite well-known in solid-state physics, such mechanisms have hardly been explored in DP modeling. Among these, modulational instability (MI) is long known to be responsible for energy localization in nonlinear lattices [44] and has been anticipated [45], [46] as an intermediate step between the linear (phonon) and strongly nonlinear (breather soliton) regime, observed in solid chains [47], [48]. Inspired by experimental evidence for nonlinear dust grain vertical oscillations in the sheath region [49], we have chosen to investigate the occurrence of the MI mechanism with respect to transverse dust-lattice waves (TDLW).

Transverse (\( \sim \hat{z} \)) dust-lattice waves (TDLW) propagating in a horizontal (\( \sim \hat{z} \)) DP crystal (lattice constant \( \tau_0 \)) is governed by

\[
M \frac{d^2 \delta z_n}{dt^2} = M \omega_0^2 (2 \delta z_n - \delta z_{n-1} - \delta z_{n+1}) + F_z \quad (13)
\]
where \( \delta z_n = z_n - z_0 \) is the displacement of the \( n \)th grain (mass \( M \), charge \( q \)) around the equilibrium position \( z_0 \). We have retained, for simplicity, only interactions between first neighbors. The transverse oscillation “eigenfrequency” is

\[
\omega_0^2 = -(q/M_r)\delta z(x) \frac{\partial^2 \Phi(x)}{\partial x^2} |_{x = x_0},
\]

where \( \Phi(x) \) is the intergrain interaction potential \((x = |x_i - x_j|)\); e.g., for a Debye–Hückel potential form \( \Phi(x) = \frac{q^2}{4\pi\varepsilon_0 x} \). The intercept \( \omega_0^2 = (q^2/M_r)\delta z(x) \frac{\partial^2 \Phi(x)}{\partial x^2} \), one has \( \omega_0^2 = (q^2/M_r)\delta z(x) \frac{\partial^2 \Phi(x)}{\partial x^2} \) (note that more sophisticated potential forms may be employed, taking into account ion flow in the sheath region [50]). The vertical force of the right-hand side \( F_z = F_{z_0} - Mg \) is the result of gravity and the upward force \( F_{z_0} = qE(z) \) due to the vertical electric field \( E(z) = -\varepsilon_0 V(z)/\delta z \). The potential \( V(z) \), in principle accounting for sheath and wake fields, the latter generated by downward ion flow [51], can be developed around the equilibrium position \( z_0 \) as \( V(z) \approx V(z_0) + V(\phi)(\delta z)^2 + (1/6) V(\phi)(\delta z)^3 + \cdots \); obviously, \( V(\phi) \equiv (\partial^2 V(z)/\partial z^2) |_{z = z_0} \). The resulting electric force \( F_{z_0} = \frac{q}{\varepsilon_0}(\partial V(z)/\partial z)(\delta z)^2 \approx \frac{q}{\varepsilon_0}(\partial V(z)/\partial z)(\delta z)^2 \) now involves: a zeroth-order term, balancing gravity at \( z_0 \), a linear (restoring) term \( \gamma(1)(\delta z)^2 \equiv \frac{q}{\varepsilon_0}(\partial V(z)/\partial z)(\delta z)^2 \equiv -M_0^2 \delta z \), and a series of higher-order (nonlinear) terms. The gap frequency \( \omega_g \) may either be evaluated from first principles [13] or determined experimentally [49]. Typically, \( \omega_g/2\pi \approx 20 \) Hz in laboratory experiments.

By linearizing and considering \( \delta z_n = A_n \exp[i(kn\tau_0 - \omega t)] + c.c. \), we obtain the TDLW dispersion relation

\[
\omega^2 = \omega_0^2 - 4\omega_0^2 \sin^2 \frac{kn\tau_0}{2}, \tag{14}
\]

cerning the linear regime, since it is sufficiently covered in the references. Let us now see what happens if the nonlinear terms are retained.

For simplicity, we shall limit ourselves to the continuum limit, considering a wavelength \( \lambda \) significantly larger than the inter-grain distance \( \tau_0 \) (e.g., \( k\tau_0 \ll 1 \)). The acceleration \( F_z/M \) in (13), which now describes a continuous function \( \delta z = u(x,t) \), is

\[
F_z/M = -\omega_0^2 u - \alpha u^2 - \beta u^3.
\]

The nonlinearity coefficients \( \alpha, \beta \) are related to the anharmonicity of the electric potential:

\[
\alpha = -\gamma(\phi)/M \equiv qV(\phi)/(2M), \beta = -\gamma(\phi)/M \equiv qV(\phi)/(6M).
\]

Considering small displacements \( u = eu_1 + e^2u_2^2 + \cdots \) and employing the reductive perturbation technique described above, we obtain a solution of the form

\[
u(x,t) = e \left[ A_0 e^{i(kx - \omega t)} + c.c. \right] + e^2 \alpha \left[ \frac{2|A|^2}{\omega_g^2} \frac{A^2}{\omega_g^2} e^{2i(kx - \omega t)} + c.c. \right] + O(e^3), \tag{15}\]

where \( \omega \) obeys the relation \( \omega^2 = \omega_0^2 - 2q^2/k^2 \) (where \( c_0 = \omega_0 \tau_0 \)). Precisely, it is (14) near \( k \approx 0 \). The slowly varying amplitude \( A = A(\xi) \) moves at the (negative) group velocity \( v_g = d\omega/dk = -2q/k/\omega \) in the direction opposite to the phase velocity. This backwave has been observed experimentally [52]. The amplitude \( A \) obeys the NLSE (5) where the “slow” variables are \( \{\xi, \tau\} \) and \( \{e(x - v_g\tau), e^2\tau\} \). The NLSE coefficients are now given by

\[
P = \omega''(k)/2 = -\sigma \omega_g^2/(2\omega^3)
\]

and

\[
Q = \frac{1}{2\omega} \left[ \frac{100\omega^2}{3\omega_g^2} - 3\beta \right] = \frac{\omega_g^2}{4\omega} \left[ \frac{5}{3} \left( \frac{V_3}{V_2} \right)^2 - \frac{V_4}{V_2} \right].
\]

One is now left with the task of evaluating \( P \) and \( Q \) for a given explicit form of \( V(z) \) and then investigating the TDL wave stability. The exact form of the potential \( V(z) \) may be obtained from experimental data fitting. For instance, in [49], the dust grain potential energy \( u(z) = qV(z) \) was reconstructed from experimental data as \( u(z) \approx M_0^2(1.00\delta z + (1/2)(\delta z)^2 - (1/3)0.5(\delta z)^3 + (1/4)0.07(\delta z)^4 + \cdots) \); see (9) therein. Upon simple comparison with our definitions above, we obtain \( V_3/V_2 = -1\text{mm}^{-1}, V_4/V_2 = 0.42\text{mm}^{-2} \), so \( Q \) is positive. Now see that \( P < 0 \), given the parabolic form of \( \omega(k) \) close to \( k = 0 \) (continuum limit). Therefore, the transverse oscillation considered in [49] would propagate as a stable wave, for large wavelength values \( \lambda \). However, for shorter wavelengths, e.g., in the general (discrete) case, the coefficient \( P = \omega''(k)/2 \) changes sign at some critical value of \( k \) (the zero-dispersion point) in the first Brillouin zone; see (14). So, \( P \) becomes positive and then so does the product \( PQ \); the TDL wave may thus be potentially unstable for higher wavenumbers.

Since the TDL wave amplitude evolution is described by the NLSE (5), the existence of either pulse-shaped localized solutions (bright envelope solitons, for \( PQ > 0 \)) or dark/gray \((PQ < 0)\) type [42], [43] may be anticipated; their analytical form is given above. The former (continuum breathers) may
occur and propagate in a DP crystal if a sufficiently short wavelength is chosen (so that $PQ > 0$). The latter (holes), yet apparently privileged in the continuum limit, are rather physically irrelevant in this (infinite chain) model, since they correspond to infinite energy stored in the lattice. These predictions may, in principle, be checked experimentally.

V. CONCLUSION

Summarizing our results, we have seen the following.

1) Electrostatic waves propagating in a complex plasma are subject to modulational instability. 
2) Allowing for an oblique wave amplitude modulation strongly affects the conditions for instability to occur; regions which are stable to a parallel modulation may become unstable when subject to an oblique modulation and vice versa. 
3) Large-angle $\theta$ modulation seems to have a stabilizing effect, while small-to-medium $\theta$ values rather promote instability. 
4) Envelope localized excitations may form and propagate in a dusty plasma; modulationally stable (unstable) values of $(k, \theta)$ sustain solitary waves of the bright (dark) type i.e., potential humps (dips), the type and characteristics of which depend on $k, \theta$ via the NLS equation coefficients. 
5) In comparison to the dust-free (e-i) plasma case, the presence of positive (negative) dust rather enhances (impedes) stability and slightly favors dark (bright) type excitations, e.g., holes (pulses).

These results may be confirmed by appropriate experiments and/or numerical investigations (beyond the scope of this paper).

Our aim has been to put forward a generic model for the study of DP mode-related amplitude modulation and harmonic generation. We have studied the modulational instabilities and associated envelope soliton formation. Besides modulational instabilities, the dust-ion-acoustic and transverse dust lattice waves could be subject to a decay instability in which a pump wave decays into a daughter wave and an ultralow-frequency (ULF) dust acoustic wave [53] or, presumably, a longitudinal ULF dust lattice wave, respectively. Thus, in the decay interaction, the lower sideband and the ULF waves are resonant, contrary to nonresonant zero-frequency modulating perturbations producing modulational instabilities. Clearly, the underlying physics of the decay and modulational instabilities are different and they will appear in different parameter regimes. Focusing on the qualitative aspects of this paper, we have ignored effects such as dust charge fluctuations and the ion drag force. It should, however, be noted that dust charge fluctuations usually produce non-Landau damping [1] of DA and DIA waves, leading to the formation of nonenvelope DA and DIA shocks [54], [55]. On the other hand, the ion drag force acting on the dust grains can produce linear instabilities [1] of DA and DIA waves depending on the dust grain radius. However, in the present paper, we have dealt with stable DA, DIA, and transverse dust lattice waves, which undergo nonlinear instabilities due to self-interactions of carrier waves.

REFERENCES


