Plasma diffusion and relaxation in a magnetic field

I. Kourakis a,*, A. Grecos b

a Association Euratom, Etat Belge, C.P. 231 Physique Statistique et Plasmas, Université Libre de Bruxelles (ULB), Boulevard du Triomphe, B-1050 Brussels, Belgium
b Euratom–Hellenic Republic Association, Laboratory of Fluid Mechanics, University of Thessaly, Athinon Avenue, Pedion Areos, GR 383 34 Volos, Greece

Abstract

A Fokker–Planck-type kinetic equation modeling a test-particle weakly interacting with an electrostatic plasma, in the presence of a magnetic field \( B \), is analytically solved. Explicit expressions are obtained for variable moments and particle density as a function of time. Diffusion in space is classical, characterized by a particle MSD varying as \(~B^2\) , in agreement with previous results.

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The standard kinetic-theoretical treatment of electrostatic plasmas is based on Landau-type equations [1], describing the evolution of a distribution function (df) \( f(v; t) \) in velocity space. This description needs to be modified in case of a non-uniform df and/or in the presence of an external force field. In previous work [2], a Fokker–Planck-type kinetic equation (FPE) was derived from first principles for a test-particle (charge \( q \), mass \( m \)) weakly interacting with a plasma embedded in a uniform magnetic field \( B \). This equation, describing the evolution of the df \( f(x, v; t) \) in phase space \( \Gamma = \{x, v\} \), has the form:

*Corresponding author. Also at: Faculté des Sciences Appliquées, ULB, Physique Générale C.P. 165/81, Avenue F.D.Roosevelt 49, B-1050 Brussels, Belgium.

E-mail addresses: ikouraki@ulb.ac.be (I. Kourakis), agrecos@mie.uth.gr (A. Grecos).
\[
\frac{\partial f}{\partial t} + \mathbf{v} \frac{\partial f}{\partial \mathbf{x}} + \Omega(\mathbf{v} \times \mathbf{B}) \frac{\partial f}{\partial \mathbf{v}} = \left( \frac{\partial^2}{\partial v_x^2} + \frac{\partial^2}{\partial v_y^2} + \frac{\partial^2}{\partial v_z^2} \right) (D_{\perp} f) + \frac{\partial^2}{\partial v_x^2} (D_{\parallel} f) + 2\Omega^{-1} \left( \frac{\partial^2}{\partial v_x \partial y} - \frac{\partial^2}{\partial v_y \partial x} \right) (D_{\perp} f)
\]
\[
+ \Omega^{-2} (Q + D_{\perp}) \left( \frac{\partial^2}{\partial v_x^2} + \frac{\partial^2}{\partial v_y^2} \right) f - \frac{\partial}{\partial v_x} (\mathcal{F}_x f) - \frac{\partial}{\partial v_y} (\mathcal{F}_y f)
\]
\[
- \frac{\partial}{\partial v_z} (\mathcal{F}_z f) + \Omega^{-1} \mathcal{F}_y \frac{\partial f}{\partial v_x} - \Omega^{-1} \mathcal{F}_x \frac{\partial f}{\partial v_y}
\]
(1)

where \( \Omega = qB/mc \) is the cyclotron frequency; \( \mathbf{B} = \mathbf{B}/B \) is the unit vector \( \mathbf{B} \); \( \partial f/\partial z = 0 \) by assumption. Note the appearance of the spatial diffusion term in the right-hand side (rhs), in fact absent from most previous studies.

In principle, one's aim should be to obtain an exact solution for \( f(t) \); one would thus be able to trace the evolution of variable moments in time, as well as their dependence on the magnetic field \( \mathbf{B} \). However, an exact analytical treatment is not possible, since all coefficients entering the collision term (rhs) are complicated functions of particle velocity \( \mathbf{v} \); in addition, they depend on the magnitude of the external magnetic field \( \| \mathbf{B} \| \). However, a numerical study of the coefficients in terms of physical parameters shows that there exists a region where the diffusion coefficients \( D_\parallel \) are practically constant (i.e., independent of \( \mathbf{v} \)) while friction terms \( \mathcal{F}_\parallel \) are linear in \( \mathbf{v} \) (throughout this text \( \dagger \) denotes either \( \perp \) or \( \| \), referring to quantities perpendicular or parallel, respectively, to \( \mathbf{B} \)). In specific, this is true for a low velocity value (compared to the thermal velocity): intuitively speaking, this is close to the standard Langevin picture of a (slow) heavy particle randomly interacting with (faster) light particles surrounding it.

Setting \( D_\parallel = \text{constant} \), \( \mathcal{F}_\parallel = \gamma \mathbf{v} \), Eq. (1) may be rearranged into the standard form of a multivariate (6d) FPE (see e.g. (VIII.6.1) in [4]). Both diffusion and drift matrices appearing in this FPE, say \( \mathbf{D} \) and \( \mathbf{A} \) respectively (in fact \( 6 \times 6 \) square matrices), are directly derived from (1) and will be omitted here, for brevity. Note that \( \mathbf{D} \) is symmetric and positive definite, as expected. Retain the equilibrium condition: \( \gamma \equiv m/T_\parallel = 2\beta_\parallel D_\parallel \) which is necessary and sufficient in order for the Maxwellian state: \( f_{\text{eq}}(\mathbf{v}) = f_{\text{eq}}(0) e^{-\beta_\parallel v_x^2} e^{-\beta_\parallel v_y^2} \) to cancel the rhs in (1). Given these considerations, which define a multi-dimensional Ornstein-Uhlenbeck process, Eq. (1) may be solved for \( f(t) \) via a Green function method. Furthermore, since it describes a Gaussian process, an exact theory exists for the calculation of variable mean values and covariances [4]. The calculation, involving multiple integrations in the \( 6 \gamma \)-space variables, is rather lengthy yet straightforward. This paper is a brief report of the results; the lengthy details are exposed elsewhere [3b].

Assuming a Maxwellian initial distribution of the form:

\[
f_0(\mathbf{x}, \mathbf{v}) = \frac{1}{\pi^{3/2}} \beta_\parallel \beta_\perp^{1/2} e^{-\beta_\parallel v_x^2} e^{-\beta_\perp v_y^2} \delta(\mathbf{x})
\]
(2)

\( (\beta_\parallel = m/2T_\parallel) \) we obtain the (time-dependent) df:

\[
f(\mathbf{x}, \mathbf{v}; t) = \frac{\tilde{\beta}_\parallel \tilde{\beta}_\perp^{1/2} \tilde{\beta}_\parallel^{(X)}}{\pi^{5/2}} e^{-\beta_\parallel v_x^2} e^{-\beta_\perp v_y^2} e^{-\tilde{\beta}_\parallel^{(X)}} \mathcal{E}^2 I_0 \left( 4 \frac{\tilde{\beta}_\parallel^{(X)}}{\Omega} v_\perp \xi \right)
\]
(3)
\[ q = x^2 + y^2, \quad v^2 = v_x^2 + v_y^2 \text{ and } v_\parallel = v_z; \text{ note the definitions:} \]

\[ \tilde{\beta}_\parallel(t) = \frac{\theta}{(1 - e^{-2y_\parallel t})\theta + e^{-2y_\parallel t}\beta_\parallel^0} \]

\[ \Xi(\tau) = \rho^2 + \left( \frac{2\beta_\perp(\tau)\zeta(\tau)}{\Omega} v_\perp \right)^2 \]

\[ \tilde{\beta}^{(X)}(\tau) = \Omega^2 \beta_\perp^0 \left\{ 1 - e^{-2\tau} + \frac{(1 - e^{-\tau})^2}{\theta} - \frac{1}{\theta} \frac{\theta^2(1 - e^{-2\tau})^2 + e^{-2\tau}(1 - e^{-\tau})^2}{\theta(1 - e^{-2\tau}) + e^{-2\tau}} + 4\beta_\perp^0 Qt \right\}^{-1} \]

\[ \zeta(\tau) = \frac{1}{\beta_\perp^0} \frac{1}{2\theta} \left\{ \theta^2(1 - e^{-2\tau})^2 + e^{-2\tau}(1 - e^{-\tau})^2 \right\}^{1/2} \]

where \( \theta = \beta_\parallel/\beta_\parallel^0 = T_\parallel^0/T_\parallel \text{ and } \tau = \gamma_\parallel t. \) We see that the velocity distribution will relax to the equilibrium state anticipated above, as physically expected, since: \( \lim_{t \to \infty} \tilde{\beta}_\parallel(\tau) = \gamma_\parallel/(2D_\parallel) \equiv \beta_\parallel^0 = m/(2T_\parallel^0), \) while spatial distribution will exhibit a classical diffusive behaviour, under the influence of collisions; check that, for \( \tau \gg \gamma^{-1} : \tilde{\beta}_\perp^{(X)}(\tau) \) behaves as \( \approx \Omega^2 \beta_\perp^0/(1 + 4\beta_\perp^0 Qt). \) The same qualitative result holds if the spatial distribution is assumed to be Maxwellian in space (i.e. not localized, cf. (2)) at \( t = 0; \) details are omitted here [3b] (Fig. 1).

We may calculate an exact expression for the mean particle density in space: \( n(x, t) = \int dvf(x, v, t). \) Considering an initial condition of the form: \( f_0(x, v) = \delta(x - x_0)\delta(v - v_0), \) we obtain the form:
\[
n(\rho, t) = \frac{1}{2\pi L^2(t)} e^{-\rho^2/2L^2(t)} e^{-R_0^2(t)/2L^2(t)} I_0 \left( \frac{R_0(t)\rho}{L^2(t)} \right) (5)
\]

(for simplicity the direction \( \parallel \) to the field was neglected and the gyro-phase \( \phi \) was averaged out)

where:

\[
L^2(t) = \Omega^{-2} \left[ \frac{T_0^2}{m} (1 - e^{-2\gamma_{\perp}t}) + 4Qt \right] (6)
\]

and \( R_0(t) = (1 + e^{-2\gamma_{\perp}t} - 2e^{-\gamma_{\perp}t} \cos \Omega t)^{1/2} v_0^0 / \Omega \); see that \( R_0 \approx v_0^0 / \Omega \equiv \rho_L^0 \) after a while. For zero initial velocity, \( R_0 = 0 \) so a pure Gaussian profile is recovered. On the other hand, for a higher \( v_0^0 \), the distribution spreads in space, as expected. See that \( L^2(t) \) asymptotically behaves as \( \sim t \); we therefore exactly recover the classical diffusion mechanism anticipated earlier. Furthermore, earlier results predicting diffusion dependence on the field as \( \sim B^{-2} \) are thus confirmed [5].

Finally, the (symmetric) covariance matrix \( \langle \langle y_k y_l \rangle \rangle = \langle \langle y_k \rangle \langle y_l \rangle \rangle = \Xi_{kl} \) \( \mathbf{y} = (x, y, z, v_x, v_y, v_z) \) can be evaluated via the same formalism [4]. For velocity covariances we obtain:

\[
\Xi_{44} = \langle \langle v_x v_x \rangle \rangle = \Xi_{55} = \langle \langle v_y v_y \rangle \rangle = \frac{T_{eq}^0}{m} (1 - e^{-2\gamma_{\perp}t})
\]

(set \( \parallel \) instead of \( \perp \) for \( \Xi_{66} = \langle \langle v_z v_z \rangle \rangle \)); position elements are:

\[
\Xi_{11} = \langle \langle xx \rangle \rangle = \Xi_{22} = \langle \langle yy \rangle \rangle = L^2
\]

(see (6)),

\[
\Xi_{33} = \langle \langle zz \rangle \rangle = \frac{D_{\parallel}}{\gamma_{\parallel}^2} \left( 2\gamma_{\parallel} t - 3 + 4e^{-\gamma_{\parallel}t} - e^{-2\gamma_{\parallel}t} \right)
\]

cross-\( v \)-x elements are:

\[
\Xi_{42} = \Xi_{51} = \langle \langle v_x y \rangle \rangle = \langle \langle v_y x \rangle \rangle = \Omega^{-1} \frac{T_{eq}^0}{m} (1 - e^{-2\gamma_{\perp}t})
\]

\[
\Xi_{63} = \langle \langle v_z z \rangle \rangle = D_{\parallel} \left( \frac{1 - e^{-\gamma_{\parallel}t}}{\gamma_{\parallel}} \right)^2
\]

(also their symmetric ones); all remaining elements are zero.

In conclusion, the analytical treatment of the kinetic equation (1) in an Ornstein-type approximation reveals a classical diffusive behaviour in space and allows for an exact study of plasma relaxation as well as moment evolution in time. A more detailed report is in preparation.

References